Math 6702, Assignment 3

Introduction

1. (Exercise 27) Find the image of the unit disk $U = \{x + iy : x^2 + y^2 < 1\}$ under the complex sine function.

2. Find the image of the unit disk $U = \{x+iy : x^2+y^2 < 1\}$ under the complex exponential function $f(z) = e^z$.

3. Let $\alpha \in (0, 1)$. Show that if $f : \mathbb{R} \to \mathbb{R}$ and there is a constant c such that f satisfies
$$|f(b) - f(a)| \le c|b - a|^\alpha \qquad \text{for every } a, b \in \mathbb{R}, \tag{1}$$
then f is continuous.

4. Let $\alpha, \gamma \in (0, 1)$ with $\gamma > \alpha$. If $f : \mathbb{R} \to \mathbb{R}$ and there is a constant C such that f satisfies
$$|f(b) - f(a)| \le C|b - a|^\gamma \qquad \text{for every } a, b \in \mathbb{R},$$
then is it necessarily true that (1) holds?

§4.3-4 Differential Approximation

5. (4.4.11-13) The first order Taylor expansion of $u \in C^1(\mathbb{R}^n)$ at $\mathbf{x}_0 \in \mathbb{R}^n$ is given by
$$P_1(\mathbf{x}) = u(\mathbf{x}_0) + \sum_{|\beta|=1} D^\beta u(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^\beta.$$

   (a) Compute the **error** $\epsilon = u(\mathbf{x}) - P_1(\mathbf{x})$ in the first order Taylor expansion to obtain a formula
   $$u(\mathbf{x}) - u(\mathbf{x}_0) = L(\mathbf{x} - \mathbf{x}_0) + \epsilon$$
   where $L : \mathbb{R}^n \to \mathbb{R}$ is a linear function. (Write down the formula for L and verify that L is linear.)

   (b) L is also called the **differential** of u and is denoted $du : \mathbb{R}^n \to \mathbb{R}$, and
   $$u(\mathbf{x}_0) + du(\mathbf{x} - \mathbf{x}_0)$$
   is called the **differential approximation** for $u(\mathbf{x})$ at $\mathbf{x}_0$. Find the differential approximation for
   $$\sqrt{(4.98)^2 - (3.03)^2}$$
   using the fact that $\sqrt{5^2 - 3^2} = 4$.

   (c) Find the differential approximation of length of the diagonal of the rectangular solid
   $$\begin{aligned}&[-1.005, 1.005] \times [-1.01, 1.01] \times [-0.505, 0.505]\\ &\qquad = \{(x, y, z) \in \mathbb{R}^3 : |x| < 1.005,\ |y| \le 1.01,\ |z| \le 0.505\}\end{aligned}$$
   using the fact that the diagonal of the rectangular solid $[-1, 1] \times [-1, 1] \times [-0.5, 0.5]$ is
   $$\sqrt{2^2 + 2^2 + 1^2} = 3.$$