Assignment 3 = Exam 1:
Differential Equations
Due Friday, February 7, 2025

John McCuan

January 21, 2025

Problem 1 (density and the hanging slinky) It was very nicely pointed out in class that the "density of coils" is not constant in a hanging slinky.

(a) Given an initial equilibrium height h_0 of a slinky (sitting on a table for example) define a **constant linear mass density** ρ_0 in terms of the total mass M of the slinky and the height h_0.

(b) Let $\sigma : [0, h_0] \to [0, \infty)$ denote a model measurement (stretch) function based on the height h_0 for the hanging slinky as described in my solution of Problem 1 of Assignment 1. Find a **linear mass density function** $\rho : [0, h_0] \to [0, \infty)$ giving/modeling the mass density at a material point in the hanging slinky corresponding to the stretch value $\sigma(h)$ for $0 \le h \le h_0$. Give your answer in terms of ρ_0 and σ. Hint: Let $\delta > 0$ and consider the portion of the hanging slinky between $X_3(h - \delta) = -\sigma(h - \delta)$ and $X_3(h + \delta) = -\sigma(h + \delta)$. How long is this portion of the hanging slinky and what is its mass?

(c) Formulate a notion of "density of coils" $\rho_c : [0, h_0] \to [0, \infty)$ for the hanging slinky and find a formula for ρ_c in terms of ρ.

Recall the existence and uniqueness theorem for a single ordinary differential equation (ODE):

Theorem 1 If $(t_0, y_0) \in (a,b) \times (c,d)$ and $F : (c,d) \times (a,b) \to \mathbb{R}$ satisfies $F \in C^0((c,d) \times (a,b))$ and

$$\frac{\partial F}{\partial y} \in C^0((c,d) \times (a,b)),$$

then there is some $\delta > 0$ for which the initial value problem (IVP)

$$\begin{cases} y' = F(y,t), & t_0 - \delta < t < t_0 + \delta \\ y(t_0) = y_0, \end{cases} \tag{1}$$

has a unique solution $y \in C^1(t_0 - \delta, t_0 + \delta)$.

Problem 2 (finite time blow-up) Consider the IVP

$$\begin{cases} y' = y^2 \\ y(3) = 7. \end{cases} \tag{2}$$

(a) Explain carefully and clearly what Theorem 1 tells you (to expect) about this problem. Clearly identify t_0, y_0 and the intervals (a,b) and (c,d).

(b) According to the statement of Theorem 1, list the quantities upon which you might expect the number δ to depend.

(c) Solve the IVP

$$\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y_0 and t_0 are given real numbers.

Problem 3 Let F satisfy the conditions of Theorem 1. If $T_1 < T_2$ and the IVP

$$\begin{cases} y' = F(y,t) \quad T_1 < t < T_2 \\ y(t_0) = y_0 \end{cases} \tag{4}$$

has a unique solution $y \in C^1(T_1, T_2)$ and for every $\delta > 0$ neither of the initial value problems

$$\begin{cases} y' = F(y,t) \quad T_1 - \delta < t < T_2 \\ y(t_0) = y_0 \end{cases} \qquad \text{or} \qquad \begin{cases} y' = F(y,t) \quad T_1 < t < T_2 + \delta \\ y(t_0) = y_0 \end{cases}$$

has a unique solution, then we say (T_1, T_2) is a **maximal interval for existence and uniqueness**.

(a) According to an application of Theorem 1 to the problem (3) list the quantities upon which you might expect the maximal interval for existence to depend. Hint: Your list should be shorter than the list you gave in part **(b)** of Problem 1. Which quantities are missing?

(b) Find the maximal interval of existence for the IVP (3).

Problem 4 (existence and uniqueness) Draw a picture illustrating the assertion of Theorem 1. Include a representation of the intervals (a, b) and (c, d), the point (t_0, y_0), and (the role played by) the number δ.

Recall the sweeping generalization of Theorem 1 which says roughly that any reasonable ODE always has a unique local solution:

Theorem 2 If

(i) Ω is an open subset of $\mathbb{R}^n$,

(ii) $(t_0, \mathbf{p}) \in (a, b) \times \Omega$, and

(iii) $\mathbf{F} = (F_1, F_2, \ldots, F_n) : \Omega \times (a, b) \to \mathbb{R}^n$ satisfies $\mathbf{F} \in C^0(\Omega \times (a, b) \to \mathbb{R}^n)$ and

$$\frac{\partial F_i}{\partial y_j} \in C^0(\Omega \times (a, b)) \qquad \text{for} \qquad i, j = 1, 2, \ldots, n,$$

then there is some $\delta > 0$ for which the initial value problem (IVP)

$$\begin{cases} \mathbf{y}' = F(\mathbf{y}, t), & t_0 - \delta < t < t_0 + \delta \\ \mathbf{y}(t_0) = \mathbf{p}, \end{cases} \tag{5}$$

has a unique solution $\mathbf{y} = (y_1, y_2, \ldots, y_n) : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n$ satisfying[1] $y_j \in C^1(t_0 - \delta, t_0 + \delta)$ for $j = 1, 2, \ldots, n$.

Problem 5 What does Theorem 2 tell you about the **second order** IVP

$$\frac{u''}{(1 + u'^2)^{3/2}} = u, \qquad u(x_0) = u_0, \qquad u'(x_0) = u_0'$$

where $x_0, u_0, u_0' \in \mathbb{R}$? Hint: Set $y_1 = u$ and $y_2 = u'$.

[1] We denote this regularity conclusion also by writing simply $\mathbf{y} \in C^1((t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n)$.

Problem 6 Consider the autonomous system of first order ordinary differential equations

$$\mathbf{x}' = \mathbf{x}^{\perp} \tag{6}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^{\perp} = (-x_2, x_1)$ and the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{\perp}, & t \in \mathbb{R} \\ \mathbf{x}(0) = (1, 0). \end{cases} \tag{7}$$

(a) Find the general solution of (6).

(b) Plot the **image** $\{(x_1(t), x_2(t)) : 0 \leq t \leq 3\pi/2\}$ of the solution of (7).

(c) Plot the **graph** $\{(t, x_1(t), x_2(t)) : t \in \mathbb{R}\}$ of the solution of (7).

Recall the following existence and uniqueness theorem for ODEs:

Theorem 3 (existence and uniqueness for linear ODEs) If $A = (a_{ij}) : (a, b) \to \mathbb{R}^{n \times n}$ is a matrix valued function satisfying $a_{ij} \in C^0(a, b)$ for $i, j = 1, 2, \ldots, n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) : (a, b) \to \mathbb{R}^n$ is a vector valued function satisfying $b_j \in C^0(a, b)$ for $j - 1, 2, \ldots, n$, then for each $(t_0, \mathbf{p}) \in (a, b) \times \mathbb{R}^n$ the IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y} + \mathbf{b}, & t \in (a, b) \\ \mathbf{y}(t_0) = \mathbf{p} \end{cases}$$

has a unique solution $\mathbf{y} \in C^1((a, b) \to \mathbb{R}^n)$.

Problem 7 (linear ODEs)

(a) In what ways is Theorem 3 “weaker” than Theorem 2?

(b) In what ways is Theorem 3 “stronger” than Theorem 2?

Problem 8 (linear ODEs) Consider the linear homogeneous IVP

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{0} \in \mathbb{R}^2.$$

(a) What is the maximal interval of existence for this IVP?

(b) Solve this IVP.

There is no sweeping theory of existence and uniqueness for partial differential equations (PDE). Most people are introduced to the subject through consideration of particular examples of PDEs. At least this is a way to get some idea of what can happen if one starts to think about partial differential equations. Very often the first PDE a person considers is Laplace's equation

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Given an open set $\Omega \subset \mathbb{R}^n$, a natural set of functions on which to consider the Laplace operator Δ and this PDE is $C^2(\Omega)$. Then we can write

$$\Delta : C^2(\Omega) \to C^0(\Omega)$$

and the Laplacian Δ is linear. This is not a bad choice of a place to start.

One source for a rich family of solutions of $\Delta u = 0$ when $\Omega \subset \mathbb{R}^2$ is the collection of **complex differentiable functions**. See Problem 7 of Assignment 2.

Problem 9 Consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$.

(a) Write $z = x+iy$ and find the real and imaginary parts $u, v \in C^\infty(\mathbb{R}^2)$ of $f = u+iv$.

(b) Compute Δu and Δv.

Problem 10 (hanging slinky) Interpret your relation from part **(b)** of Problem 1 above involving the stretch function σ and the linear density function ρ as an ordinary differential equation with two unknown functions σ and ρ.

(a) What else would you need in order to apply an existence and uniqueness theorem for ODEs to potentially determine the stretch σ and the linear mass density ρ?

(b) What natural restriction or constraint should apply to the derivative σ' of the stretch function and why?

(c) With a view to applying an existnce and uniqueness theorem for ODEs to some ODEs for ρ and σ, what can you say about the potential boundary and/or initial values.