Assignment 3 = Exam 1:
Ordinary Differential Equations (and other topics)
Due Wednesday, February 8, 2023

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January 16, 2023

Problem 1 Consider the autonomous system of first order ordinary differential equations

$$\mathbf{x}' = \mathbf{x}^{\perp} \tag{1}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^{\perp} = (-x_2, x_1)$ and the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{\perp}, & t \in \mathbb{R} \\ \mathbf{x}(0) = (1, 0). \end{cases} \tag{2}$$

(a) Find the general solution of (1).

(b) Plot the **image** $\{(x_1(t), x_2(t)) : 0 \le t \le 3\pi/2\}$ of the solution of (2).

(c) Plot the **graph** $\{(t, x_1(t), x_2(t)) : t \in \mathbb{R}\}$ of the solution of (2).

Problem 2 Find a single (second order) ordinary differential equation equivalent to the system (1) in Problem 1 above by setting $y = x_1$.

Problem 3 Consider the nonautonomous system of first order ODEs

$$\begin{cases} x_1' = -x_2 \sec^2 t \\ x_2' = x_1 \sec^2 t \\ u' = \sec^2 t \end{cases} \tag{3}$$

and the initial value problem

$$\begin{cases} x_1' = -x_2 \sec^2 t, & x_1(0) = 1 \\ x_2' = x_1 \sec^2 t, & x_2(0) = 0 \\ u' = \sec^2 t, & u(0) = 0. \end{cases} \tag{4}$$

(a) Find the general solution of (3) on the interval $|t| < \pi/2$.

(b) Plot the **image** $\{(x_1(t), x_2(t), u(t)) : -\pi/2 < t \leq \pi/2\}$ of the solution of (4).

(c) Plot the projection $\{(x_1(t), x_2(t)) : 0 \leq t \leq \tan^{-1}(3\pi/2)\}$ of the image of the solution of (4).

Any time the integral of a function $f : \mathbb{R} \to \mathbb{R}$ makes sense, let us write

$$\int_{\mathbb{R}} f \tag{5}$$

for the value of **the integral of** f on all of $\mathbb{R}$. This is a little more general that what you have probably seen before, but it includes some cases you know. For example, (you know that) if $f \in C^0(\mathbb{R})$ and

$$\lim_{R\to\infty} \int_{-R}^{R} f(t)\, dt = I \in \mathbb{R},$$

then

$$\int_{-\infty}^{\infty} f(t)\, dt = I = \int_{\mathbb{R}} f.$$

Problem 4 (test functions) Let $C_c^0(\mathbb{R})$ denote the subspace of $C^0(\mathbb{R})$ of **continuous functions with compact support**, that is,

$$C_c^0(\mathbb{R}) = \{\phi \in C^0(\mathbb{R}) : \text{there exists some } R > 0 \text{ with } \phi(x) = 0 \text{ for } |x| \geq R\}.$$

Note that

$$\int_{\mathbb{R}} \phi$$

makes sense for every $\phi \in C_c^0(\mathbb{R})$ and

$$\int_{\mathbb{R}} u\phi$$

makes sense for every $u \in C^0(\mathbb{R})$ and $\phi \in C_c^0(\mathbb{R})$.

Let $C_c^1(\mathbb{R})$ denote the subspace of $C^1(\mathbb{R})$ of **continuously differentiable functions with compact support**, that is,

$$C_c^1(\mathbb{R}) = \{\phi \in C^1(\mathbb{R}) : \text{there exists some } R > 0 \text{ with } \phi(x) = 0 \text{ for } |x| \geq R\}.$$

(a) (characteristic function) Given a set $A \subset \mathbb{R}$, consider $\chi_A : \mathbb{R} \to \mathbb{R}$ by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Draw the graph of $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = [g(x) - R]\chi_{(-R,R)}(x)$ where g is the absolute value function and compute

$$\int_{\mathbb{R}} \phi.$$

(b) Given $x_0 \in \mathbb{R}$ and $r > 0$, construct a function $\phi \in C_c^1(\mathbb{R})$ having the following properties

(i) $\phi \geq 0$.

(ii) $\phi(x) > 0$ if and only if $|x - x_0| < r$.

(iii) $\phi(x_0 - t) = \phi(x_0 + t)$ for $t \in \mathbb{R}$.

(c) Show that if $f \in C^0(\mathbb{R})$ and

$$\int_{\mathbb{R}} f\phi = 0 \qquad \text{for every } \phi \in C_c^1(\mathbb{R}),$$

then $f(x) = 0$ for every $x \in \mathbb{R}$.

We can call this the **fundamental lemma of test functions**.[1]

Problem 5 (weak derivatives) We say $v : \mathbb{R} \to \mathbb{R}$ is a **weak derivative** of $u : \mathbb{R} \to \mathbb{R}$ if

$$-\int_{\mathbb{R}} u\phi' = \int_{\mathbb{R}} v\phi \qquad \text{for all } \phi \in C_c^1(\mathbb{R}), \tag{6}$$

(and all the integrals in (6) make sense).

(a) Find a weak derivative $g' : \mathbb{R} \to \mathbb{R}$ for the absolute value function $g : \mathbb{R} \to \mathbb{R}$ with the following properties.

(L) The restriction

$$g'\big|_{(-\infty,0)}$$

of g' to the open interval $(-\infty, 0)$ is in $C^0(-\infty, 0)$.

(R) The restriction

$$g'\big|_{(0,\infty)}$$

of g' to the open interval $(0, -\infty)$ is in $C^0(0, +\infty)$.

(b) Show that if v is any weak derivative of the absolute value function g and

$$v\big|_{\mathbb{R}\backslash\{0\}} \in C^0(\mathbb{R}\backslash\{0\}),$$

[1] In some form, it is also called the **fundamental lemma of the calculus of variations**.

then

$$v\Big|_{\mathbb{R}\backslash\{0\}} = g'\Big|_{\mathbb{R}\backslash\{0\}}$$

where g' is the function you found in part **(a)** above.

(c) Find 10^6 other functions that are weak derivatives of the absolute value function g (which are all different from the function g' you found and different from each other).

Problem 6 Let $h : \mathbb{R} \to \mathbb{R}$ denote the Heaviside function

$$h(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 \le x < \infty \end{cases}$$

as in Assignment 1 Problem 7.

(a) Compute $-\int_{\mathbb{R}} h\phi'$ for $\phi \in C_c^1(\mathbb{R})$. Hint: Use the fundamental theorem of calculus.

(b) Show that h does not have any weak derivative v satisfying

$$v\Big|_{\mathbb{R}\backslash\{0\}} \in C^0(\mathbb{R}\backslash\{0\}).$$

Hint: Use the technique you used in part **(b)** of Problem 5 above.

Note: Sometimes we wish to extend and modify the notation in (5) in various ways. One extension is to allow more general domains of integration. If $A \subset \mathbb{R}$ and

$$\int_{\mathbb{R}} f\chi_A$$

makes sense, then we can write

$$\int_A f = \int_{\mathbb{R}} f\chi_A. \tag{7}$$

It is sometimes useful to modify the notation(s) (5) and (7) for integrals so that a **variable of integration** is specified (or emphasized) much in the same way the variable of integration is specified in Leibniz' notation for the integration of continuous functions

$$\int_a^b f(t)\,dt.$$

This may be done by writing the variable of integration in the "limit of integration" as follows:

$$\int_{t\in A} f = \int_{t\in A} f(t).$$

Notice that in the notational variant on the right, the name of the function **may not** appear:

$$\int_{t\in(0,\infty)} \left(\frac{\chi_{(0,1)}(t)}{\sqrt{t}} + \frac{\chi_{(1,\infty)}(t)}{t^2} \right) = \int_0^1 \frac{1}{\sqrt{t}}\,dt + \int_1^\infty \frac{1}{t^2}\,dt = 3.$$

Problem 7 (weak solutions) Consider the (single, nonautonomous) ordinary differential equation $x' = f(x,t)$ for $x \in C^1(\mathbb{R})$ where $f \in C^2(\mathbb{R}^2)$ is given.

Here is a definition:

Definition 1 (weak solution of a single ODE) *A function* $u \in C^0(\mathbb{R})$ *is a* **continuous weak solution** *of the ordinary differential equation* $x' = f(x,t)$ *if*

$$-\int_{\mathbb{R}} u\phi' = \int_{t\in\mathbb{R}} f(u(t),t)\phi(t) \qquad \textit{for every } \phi \in C_c^1(\mathbb{R}).$$

(a) You (should) know what $f \in C^0(\mathbb{R}^2)$ means, and you (should) know what $f \in C^1(\mathbb{R}^2)$ means. Can you guess (or find out) what $f \in C^2(\mathbb{R}^2)$ means?

(b) Find a weak solution of the ordinary differential equation

$$\frac{dy}{dt} = 2h(t) - 1 \tag{8}$$

where h is the Heaviside function.

(c) Find all continuous weak solutions of the ordinary[2] differential equation (8).

(d) Show that if u is a continuous weak solution of $x' = f(t)$, then every (other) continuous weak solution w of $x' = f(t)$ satisfies $w(x) = u(x) + c$ for some constant $c \in \mathbb{R}$. (Hint: Show every continuous weak solution of $x' = 0$ is constant.)

[2]Or maybe not so ordinary.

Problem 8 (vector space; linear algebra) In Problem 4 above I mentioned that the **spaces** $C_c^0(\mathbb{R})$ and $C_c^1(\mathbb{R})$ of test functions were **subspaces** of $C^0(\mathbb{R})$ and $C^1(\mathbb{R})$ respectively. I had in mind the following definitions from linear algebra:

Definition 2 (vector space) *A set V is a* **real vector space** *if there is an operation of* **addition** $+ : V \times V \to V$ *by* $(v, w) \mapsto v + w$*, i.e., a way to add two vectors in V to get back a (sum) vector $v + w$ in V, satisfying*

VS1 $v + w = w + v$ *and* $(v + w) + z = v + (w + z)$*, i.e., addition is commutative and associative.*

VS2 *There is some vector $\mathbf{0} \in V$, called the* **zero vector** *for which*

$$v + \mathbf{0} = v \qquad \textit{for every } v \in V.$$

The zero vector is also called the **additive identity***.*

VS3 *For every $v \in V$ there is a vector $w \in V$ such that*

$$v + w = \mathbf{0}.$$

In this case, the vector w is called the **additive inverse** *of v and is denoted by $-v$.*

In addition to addition, there is **scaling** $\mathbb{R} \times V \to V$ *by* $(a, v) \mapsto av$ *which satisfies*

VS4 $(ab)v = a(bv)$ *for all $a, b \in \mathbb{R}$ and $v \in V$.*

VS5 $1v = v$ *for all $v \in V$.*

VS6 $a(v + w) = av + aw$*. (Scaling distributes across vector addition.)*

VS7 $(a + b)v = av + bv$*. (A scaled vector distributes across a sum of scalars.)*

Definition 3 *Given a vector space V, a subset $W \subset V$ is said to be a* **subspace** *of V is W is a vector space with respect to the same operations and with the same additive identity.*

Definition 4 *Given two real vector spaces V and W, a function $L : V \to W$ is said to be* **linear** *if*

$$L(av + bw) = aL(v) + bL(w) \qquad \textit{for all } a, b \in \mathbb{R} \textit{ and } v, w \in V.$$

(a) Show $C^0(\mathbb{R})$ is a vector space.

(b) Show that given any vector space V the subset $W \subset V$ is a subspace if and only if W satisfies the following property:

For any $a, b \in \mathbb{R}$ and $v, w \in W$ there holds $av + bw \in W$.

In this case, W is said to be **closed under scaling and addition**, or simply closed under linear combinations.

(c) Show $C_c^1(\mathbb{R})$ is a subspace of $C^0(\mathbb{R})$.

(d) Given $f \in C^0(\mathbb{R})$, show $P : C_c^1(\mathbb{R}) \to \mathbb{R}$ by

$$P(\phi) = -\int_{\mathbb{R}} f\phi'$$

is linear. This linear function P is called the **weak differentiation operator**.

(e) Show $D : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ by

$$Du = \frac{du}{dx}$$

is linear. This linear function D is called the **classical differentiation operator**.

Problem 9 A function $f \in C^\infty(\mathbb{R})$, i.e., a function with derivatives $f^{(j)}$ existing for every $j = 0, 1, 2, \ldots$, is said to be **real analytic** or C^ω (read "f is C-omega") if for each $x, x_0 \in \mathbb{R}$ the series

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \tag{9}$$

converges to a real number and that real number satisfies

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j.$$

(a) Show the exponential function $\exp(x) = e^x$ is in $C^\omega(\mathbb{R})$.

(b) Find a function $u \in C^\infty(\mathbb{R})\backslash C^\omega(\mathbb{R})$.

Problem 10 There is a series construction/expansion similar to the one you know from Problem 9 above which applies to functions of several variables. Given an open set $U \subset \mathbb{R}^n$ and a function $u \in C^\infty(U)$, meaning that all partial derivatives of all orders are well-defined, the **multidimensional Taylor series** associated with u at $\mathbf{x}_0 \in U$ is defined to be

$$\sum_{j=1}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!}(\mathbf{x} - \mathbf{x}_0)^\beta. \tag{10}$$

You (most likely) do not understand (many things about) this expression. Please proceed anyway.

(a) In the expansion formula (10) the symbol $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ denotes a **multi-index**. This means $\beta_j \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ for each $j = 1, 2, \ldots, n$. The "magnitude" of the multi-index β is defined by

$$|\beta| = \sum_{j=1}^{n} \beta_j.$$

Find all the multiindices with $|\beta| = 2$ when $n = 3$.

(b) The **multi-index partial derivatives** appearing in (10) are given in the usual notation by

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}.$$

Perhaps you can see why the multi-index partial derivative notation is preferable when one is dealing with high order partial derviatives. For example, given $u = u(x, y, z)$ we can write

$$D^{(2,0,0)}u = \frac{\partial^2}{\partial x^2}.$$

Write down all the second partials of a function u when $n = 3$ in both forms as I have done.

(c) The factorial and the power appearing in (10) are defined as follows:

$$\beta! = \beta_1!\beta_2! \cdots \beta_n! \qquad \text{and} \qquad \mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}.$$

On page 192 Boas gives the second order terms of the power series expansion for a function of two variables:

$$\frac{1}{2!}\left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2\right].$$

Isolate the second order terms in (10) when $n = 2$, and show the second order terms form (10) are the same as Boas' second order terms.

(d) (Boas Problem 4.2.5) Find the Taylor expansion of $u(x, y) = \sqrt{1 + xy}$ at $(x_0, y_0) = (0, 0)$ using Boas' formula

$$u(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left((x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right)^n f\Big|_{(x_0,y_0)}$$

and using formula (10).