MATH 6702 Assignment 3
Due Monday March 8, 2021

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The following problems fall into four categories: calculus of variations, multivariable calculus, abstract spaces, and gradient flow.

Calculus of Variations

Problem 1 *(Boas 9.5.8-9) Sometimes in Newtonian particle mechanics, i.e., when using the equation* $\mathbf{f} = m\mathbf{a}$*, you find all the forces using some sort of force diagram and sum up the forces to write down the ODE. Now that you know the calculus of variations, there is a powerful method to write down forms of Newton's second law in very complicated situations, and with respect to coordinates, where it might not be so easy to determine all the forces.*

(a) *Look up* **Hamilton's principle of mechanics** *and state it in your own words using correctly the framework of the calculus of variations. Hint: You'll need to define a particular functional and mention the role played by extremals of the functional.*

(b) *Introduce appropriate time dependent variables/quantities to describe the motion of two point masses, one of which "falls" down the negative z-axis from the tip of of a cone*

$$\{(x, y, z) : z = \sqrt{x^2 + y^2}\}$$

and the other of which slides (without friction) on the surface of the cone, assuming the two point masses are connected with an inextensible (and massless) string of fixed positive length. Hint: See Example 3 of section 9.5 of Boas and use spherical coordinates for the point mass on the cone.

(c) *Use Hamilton's principle to write down the equations of motion, i.e., an appropriate initial value problem, in terms of your variables as Euler-Lagrange equations.*

(d) *Find some explicit solutions in an interesting special case—well, maybe not that interesting.*

(e) *Find and animate a numerical solution when both masses are assumed to be unit masses, the hanging mass starts from rest at the vertex of the cone, the string has length $\sqrt{2}$, and the mass on the cone starts at $(1, 0, 1)$ with velocity $(0, 1, 0)$.*

Problem 2 *(9.8.1; first integral for an autonomous Lagrangian) If $\mathcal{F}$ is a Lagrangian integral functional with Lagrangian $F = F(z, p)$, independent of the variable $x \in (a, b)$, then an extremal $u \in C^2(a, b)$ satisfies a first order ODE, called the* **first integral equation** *or* **Erdmann's equation**.

(a) *Show that if $u \in C^2(a, b)$ is a weak extremal for $\mathcal{F}$ in the situation described above, then there exists a constant $c \in \mathbb{R}$ for which*

$$u' \frac{\partial F}{\partial p}(u, u') - F(u, u') = c.$$

This is Erdmann's equation. Hint: $E : C^1(a, b) \to C^0(a, b)$ by $E[u] = u' F_p(u, u') - F(u, u')$ is called the Erdmann operator. Differentiate the Erdmann operator (with respect to x).

(b) *Consider $\sigma : C^1[a, b] \to \mathbb{R}$ by*

$$\sigma[u] = \int_a^b (u'^2 + u^2 - u)\, dx.$$

Show that for this functional there exist solutions of Erdmann's equation which are **not** *extremals for σ. What does this tell you about the use of Erdmann's equation as a first integral of the Euler-Lagrange equation?*

Problem 3 *(hanging chain; Assignment 1 Problems 3 and 9) Let a, b, y_1 and y_2 be given real numbers with $a < b$. Also, let L be a constant with*

$$L > \sqrt{(b - a)^2 + (y_2 - y_1)^2}.$$

(a) *Use the method of Lagrange multipliers to obtain an Euler-Lagrange ODE satisfied by a minimizer* $u \in \mathcal{A} = \{w \in C^2[a,b] : w(a) = y_1,\ w(b) = y_2\}$ *of the functional* $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ *by*

$$\mathcal{F}[w] = \int_a^b w\sqrt{1 + w'^2}\, dx$$

subject to the constraint

$$\mathcal{L}[w] = \int_a^b \sqrt{1 + w'^2}\, dx = L.$$

(b) *The equation given by Boas to model the shape of a hanging chain is/was*

$$(y'')^2 = k^2[1 + (y')^2].$$

Is this your Euler-Lagrange equation? If not, can you say which equation provides a better model? Are they somehow the same? Hint: You may wish to use the previous problem.

Problem 4 *(Poisson's equation) Let* $\mathcal{U}$ *be an open bounded subset of* $\mathbb{R}^2$*.*

(a) *Given a point* $\mathbf{p} \in \mathcal{U}$ *and an open ball* $B_\delta(\mathbf{p}) \subset \mathcal{U}$*, find a function* $\phi \in C_c^\infty(\mathcal{U})$ *with* $\phi \geq 0$ *and* $\phi(\mathbf{x}) > 0$ *precisely for* $\mathbf{x} \in B_\delta(\mathbf{p})$*.*

(b) *State and prove a version of the fundamental lemma of the calculus of variations for continuous functions* $f \in C^0(\mathcal{U})$*.*

(c) *Find the partial differential equation satisfied by extremals* $u \in C^2(\mathcal{U})$ *of the Dirichlet energy*

$$\mathcal{D}[u] = \int_\mathcal{U} |Du|^2$$

subject to the constraint

$$\mathcal{V}[u] = \int_\mathcal{U} u = V.$$

Multivariable Calculus

Recall that continuity of a function $f : X \to Y$ with domain a metric space X and co-domain a metric space Y is defined as follows:

A function $f : X \to Y$ is **continuous** at $x_0 \in X$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \epsilon \qquad \text{whenever } x \in B_\delta(x_0).$$

The function is said to be **continous on** X is f is continuous at each point $x_0 \in X$. In this case we write $f \in C^0(X \to Y)$. That is, $C^0(X \to Y)$ is the collection of all continuous functions from the metric space X to the metric space Y. In the special case where $Y = \mathbb{R}$ and f is a real valued function, we write simply $f \in C^0(X)$.

Problem 5 *(differentiability implies continuity) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x_0 \in (a, b)$, then f is continuous at x_0.*

Recall that the **partial derivatives** of a function $u : \mathcal{U} \to \mathbb{R}$ defined on an open set $\mathcal{U} \subset \mathbb{R}^2$ are defined at $\mathbf{x} \in \mathcal{U}$ by

$$\frac{\partial u}{\partial x}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_1) - u(\mathbf{x})}{h} \qquad \text{and} \qquad \frac{\partial u}{\partial y}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_2) - u(\mathbf{x})}{h}$$

if these limits exist. Remember that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. If the partial derivatives of a function of several variables exists, we say the function is **partially differentiable**.

Problem 6 *(partial differentiability and continuity)*

(a) *Give an example of a function $u : \mathbb{R}^2 \to \mathbb{R}$ for which both partial derivatives exist (at all points in $\mathbb{R}^2$) but the function is discontinuous at at least one point.*

(b) *A partially differentiable function $u : \mathcal{U} \to \mathbb{R}$ is said to be* **differentiable** *at $\mathbf{p} \in \mathcal{U}$ if there exists a linear function $L : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{|\mathbf{x} - \mathbf{p}|} = 0.$$

Show that a function $u : \mathcal{U} \to \mathbb{R}$ which is differentiable at $\mathbf{p}$ is continuous at $\mathbf{p}$.

(c) *Give an example of a function which is partially differentiable but not differentiable. Hint: Look at part (a) and part (b).*

(d) *If a function* $u : \mathcal{U} \to \mathbb{R}$ *is differentiable at a point* $\mathbf{p} \in \mathcal{U}$, *then the linear function* $L : \mathbb{R}^2 \to \mathbb{R}$ *given in the definition is called the differential of* u *at* $\mathbf{p}$. *Show the differential is given by the dot product with the vector of first partials:*

$$L(\mathbf{v}) = du_{\mathbf{p}}(\mathbf{v}) = \mathbf{v} \cdot \left(\frac{\partial u}{\partial x}(\mathbf{p}), \frac{\partial u}{\partial y}(\mathbf{p}) \right).$$

The vector of first partials

$$\left(\frac{\partial u}{\partial x}(\mathbf{p}), \frac{\partial u}{\partial y}(\mathbf{p}) \right)$$

in this case is called the **gradient** *or* **total derivative** *and is denoted by* $Du = Du(\mathbf{p})$. *Note that* $Du : \mathcal{U} \to \mathbb{R}^2$. *This makes* Du *a vector field on* $\mathcal{U}$.

(e) *A function* $u : \mathcal{U} \to \mathbb{R}$ *where* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^2$ *is said to be in* $C^1(\mathcal{U})$ *if the partial derivatives satisfy*

$$\frac{\partial u}{\partial x} \in C^0(\mathcal{U}) \qquad \text{and} \qquad \frac{\partial u}{\partial y} \in C^0(\mathcal{U}).$$

Show that if $u \in C^1(\mathcal{U})$, *then* u *is differentiable at each point in the open set* $\mathcal{U}$.

(f) *(directional derivatives) Given a function* $u \in C^1(\mathcal{U})$ *with* $\mathcal{U}$ *an open subeset of* $\mathbb{R}^2$ *and a unit vector* $\mathbf{u} \in \mathbb{R}^2$ *the* **directional derivative** *of* u *in the direction* $\mathbf{u}$ *at* $\mathbf{p} \in \mathcal{U}$ *is defined by*

$$D_{\mathbf{u}} u(\mathbf{p}) = \lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{u}) - u(\mathbf{p})}{h}.$$

Assume $u \in C^1(\mathcal{U})$ *as above with* $\mathbf{p} \in \mathcal{U}$ *and* $\mathbf{u}$ *a unit vector in* $\mathbb{R}^2$. *Use the chain rule to show the following*

(i) *If* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ *by* $\mathbf{r}(t) = \mathbf{p} + t\mathbf{u}$, *then*

$$D_{\mathbf{u}} u(\mathbf{p}) = \frac{d}{dt} u \circ \mathbf{r}(t)\Big|_{t=0}.$$

(ii) $D_{\mathbf{u}} u(\mathbf{p}) = Du(\mathbf{p}) \cdot \mathbf{u}$.

(i) *If* $\mathbf{r} : (a, b) \to \mathcal{U}$ *is any path in* $\mathcal{U}$ *satisfying* $\mathbf{r}(0) = \mathbf{p}$ *and* $\mathbf{r}'(0) = \mathbf{u}$, *then*

$$D_{\mathbf{u}} u(\mathbf{p}) = \frac{d}{dt} u \circ \mathbf{r}(t)\Big|_{t=0}.$$

Problem 7 *(Boas 5.2.22) Use an area integral to find the volume above the triangle with vertices* $(0, 2, 0)$, $(1, 1, 0)$, *and* $(2, 2, 0)$ *but below the graph of* $u(x, y) = xy$.

Problem 8 *(Lagrange multipliers) Let* c *be a constant, and let* $u, g \in C^1(\mathcal{U})$ *with* $\mathcal{U} \subset \mathbb{R}^2$. *If* $\mathbf{p} \in \mathcal{U}$ *is a point for which* $g(\mathbf{p}) = c$ *and we have* $u(\mathbf{p}) \leq u(\mathbf{x})$ *for all* $\mathbf{x} \in \mathcal{U}$ *with* $g(\mathbf{x}) = c$, *then show one of the two following conditions holds:*

(i) $Dg(\mathbf{p}) = \mathbf{0}$, *or*

(ii) *There exists some* $\lambda \in \mathbb{R}$ *with*

$$D(u - \lambda g)(\mathbf{p}) = \mathbf{0}.$$

Problem 9 *(weak maxiumum principle) Given a function* $u \in C^2(\mathcal{U})$ *where* $\mathcal{U}$ *is an open set in* $\mathbb{R}^2$, *define the* **Hessian matrix** *to be the matrix of second partial derivatives of* u:

$$D^2 u = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x \partial y} \\ \\ \dfrac{\partial^2 u}{\partial x \partial y} & \dfrac{\partial^2 u}{\partial y^2} \end{pmatrix}.$$

Given a unit vector $\mathbf{u} \in \mathbb{R}^2$ *and a point* $\mathbf{p} \in \mathcal{U}$, *the* **second directional derivative** *of* u *at* $\mathbf{p}$ *in the direction* $\mathbf{u}$ *is defined by*

$$D_{\mathbf{uu}} u(\mathbf{p}) = \frac{d^2}{dt^2} u(\mathbf{p} + t\mathbf{u})\Big|_{t=0}.$$

(a) *If* $u, v \in C^2(\mathcal{U})$ *with* $u(\mathbf{x}) \leq v(\mathbf{x})$ *for every* $\mathbf{x} \in \mathcal{U}$ *and* $u(\mathbf{p}) = v(\mathbf{p})$ *for some* $\mathbf{p} \in \mathcal{U}$, *then Show*

$$Du(\mathbf{p}) = Dv(\mathbf{p}) \qquad \text{and} \qquad \langle D^2 u(\mathbf{p})\mathbf{w}, \mathbf{w}\rangle \leq \langle D^2 v(\mathbf{p})\mathbf{w}, \mathbf{w}\rangle \quad \textit{for every } \mathbf{w} \in \mathbb{R}^2.$$

(b) *Show the* **second directional derivative** *in the unit direction* $\mathbf{u}$ *is given by*

$$D_{\mathbf{uu}} u(\mathbf{p}) = \langle D^2 u(\mathbf{p})\mathbf{u}, \mathbf{u}\rangle.$$

(c) *Show that if* A *is a bounded subset of* $\mathbb{R}^2$, *then* $\overline{A}$ *is bounded in* $\mathbb{R}^2$.

(d) *Let $\mathcal{U}$ be an open bounded subset of $\mathbb{R}^2$. Show that if $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ is a solution of Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{on } \mathcal{U},$$

then

$$u(\mathbf{p}) \leq \max_{x \in \partial \mathcal{U}} u(\mathbf{x}) \qquad \text{for all } \mathbf{p} \in \mathcal{U}.$$

Hint: Assume there is some $\mathbf{p}_0 \in \mathcal{U}$ with

$$u(\mathbf{p}_0) > \max_{x \in \partial \mathcal{U}} u(\mathbf{x}),$$

then consider an appropriate vertical translation of $v(x, y) = -\epsilon|\mathbf{x} - \mathbf{p}|^2$. Calculate Δv.

Gradient Flow

Problem 10 *Let $u \in C^1(\mathcal{U})$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, and define the* **gradient flow** *or* **gradient descent** *determined by u as the collection of initial value problems*

$$\begin{cases} \mathbf{x}' = -Du(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{p} \end{cases}$$

determined by the points $\mathbf{p} \in \mathcal{U}$.

(a) *Let $\mathbf{p} \in \mathcal{U}$ be fixed. Use the properties of the Euclidean inner product to find the minimum value of the directional derivative*

$$D_{\mathbf{u}} u(\mathbf{p})$$

which can be attained by taking $\mathbf{u} \in \mathbb{R}^2$ with $|\mathbf{u}| = 1$.

(b) *Find (and solve) all gradient flows determined by $u(x, y) = \sin x + \cos y$.*

Abstract Spaces Recall that we discussed metric spaces and normed spaces last semester. Here is another kind of abstract space with even more structure.

Problem 11 *(inner product space) Given a vector space V, a function*

$$\langle \cdot\,,\ \cdot \rangle : V \times V \to \mathbb{R}$$

is called a **real inner product** *if the following conditions hold*

(i) $\langle v, w\rangle = \langle w, v\rangle$ *for all $v, w \in V$. (symmetric)*

(ii) $\langle av + bw, z\rangle = a\langle v, z\rangle + b\langle w, z\rangle$ *for all $v, w, z \in V$ and $a, b \in \mathbb{R}$. (bilinearity)*

(iii) $\langle v, v\rangle \geq 0$ *with equality if and only if $v = \mathbf{0}$. (positive definite)*

In words, an inner product is a bilinear, symmetric positive definite function on a vector space. A vector space with an inner product is called an **inner product space***.*

(a) *Show every inner product space is a normed space with norm given by*

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

If you do not know (or do not remember) the definition of a normed space, you'll need to look it up.

(b) *Show the Cauchy-Schwarz inequality holds in any inner product space:*

$$|\langle v, w\rangle| \leq \|v\|\,\|w\| \qquad \text{for all } v, w \in V,$$

with equality if and only if v is parallel to w, that is to say, either $w = \mathbf{0}$ or there is some scalar λ for which $v = \lambda w$. Hint: Compute $\|v + \lambda w\|^2$ and then complete the square with respect to λ.

(c) *Show $\langle \cdot\,,\ \cdot \rangle : C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ by*

$$\langle u, v\rangle = \int_{\mathbb{R}^n} uv$$

defines an inner product on $C_c^\infty(\mathbb{R}^n)$.