Assignment 2: Solutions
Due Friday, January 31, 2025

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Problem 1 Draw the set

$$\{\mathbf{x} + h\mathbf{e}_j : |h| < \delta\} \subset \mathbb{R}^n$$

when

(a) $n = 2$, $\mathbf{x} = (2, 1)$ and $\delta = 1/3$.

(b) $n = 3$, $\mathbf{x} = (1, 0, 1/2)$ and $\delta = 1/4$.

Solution:

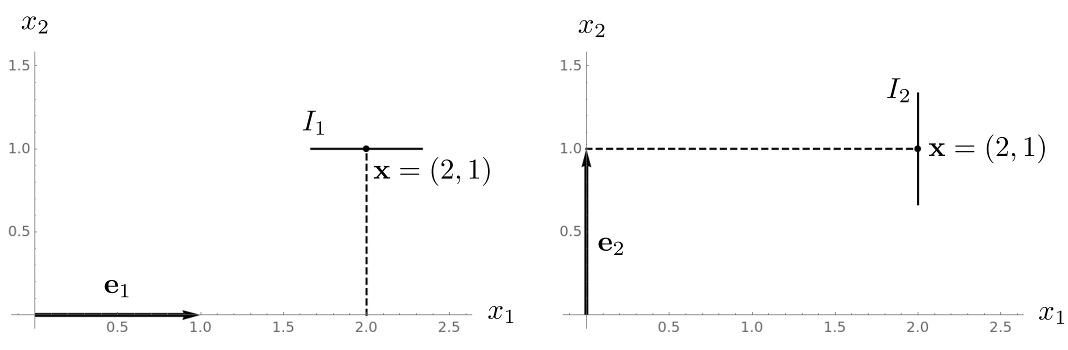

Figure 1: $I_1 = \{(2,1) + h\mathbf{e}_1 : |h| < 1/3\}$ and $I_2 = \{(2,1) + h\mathbf{e}_2 : |h| < 1/3\}$ in $\mathbb{R}^2$

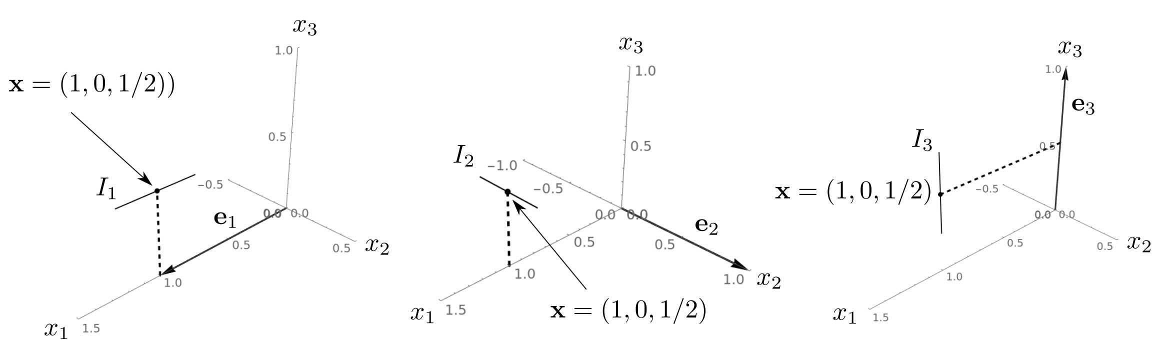

Figure 2: $I_1 = \{(1,0,1/2) + h\mathbf{e}_1 : |h| < 1/4\}$, $I_2 = \{(1,0,1/2) + h\mathbf{e}_2 : |h| < 1/4\}$, and $I_3 = \{(1,0,1/2) + h\mathbf{e}_3 : |h| < 1/4\}$ in $\mathbb{R}^3$

Problem 2 (convexity) Recall the following definition: A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2) \tag{1}$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \le t \le 1$.

Show that if f is convex, then

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{2}$$

Hint: Assume there is some $\epsilon > 0$ and a sequence of points $x_1, x_2, x_3, \dots$ with $x_j \nearrow x_0$ and $f(x_j) \le f(x_0) - \epsilon$. If you can get a contradiction out of this, it means

$$\lim_{x \nearrow x_0} f(x) \ge f(x_0).$$

What does (2) tell you about convex functions?

Solution: The details of the suggested proof may be found in my solution of Problem 4 of Assignment 1. I will give a somewhat different and more direct proof here:

I proceed to establish some inequalities. Let

$$\alpha = \frac{a + x_0}{2} \qquad \text{and} \qquad \beta = \frac{x_0 + b}{2}.$$

It is very easy to see that $a < \alpha < \beta < b$. If $\alpha < x < x_0$, then taking

$$\lambda = \frac{x - \alpha}{x_0 - \alpha}$$

I have

$$(1-\lambda)\alpha + \lambda x_0 = \frac{x_0 - x}{x_0 - \alpha}\alpha + \frac{x - \alpha}{x_0 - \alpha}x_0 = x.$$

Therefore by convexity

$$\begin{aligned} f(x) &\le (1-\lambda)f(\alpha) + \lambda f(x_0) \\ &= f(x_0) + \frac{x_0 - x}{x_0 - \alpha}[f(\alpha) - f(x_0)] \\ &= f(x_0) + \frac{f(x_0) - f(\alpha)}{x_0 - \alpha}(x - x_0). \end{aligned} \tag{3}$$

Next, let

$$\lambda = \frac{x_0 - x}{\beta - x} \qquad \text{so that} \qquad 1 - \lambda = \frac{\beta - x_0}{\beta - x} \qquad \text{and} \qquad (1-\lambda)x + \lambda\beta = x_0.$$

In this case, we have by convexity

$$f(x_0) \leq (1-\lambda)f(x) + \lambda f(\beta)$$

or

$$\begin{aligned} f(x) &\geq \frac{1}{1-\lambda}[f(x_0) - \lambda f(\beta)] \\ &= \frac{\beta - x}{\beta - x_0} f(x_0) - \frac{x_0 - x}{\beta - x_0} f(\beta) \\ &= f(x_0) + \frac{f(\beta) - f(x_0)}{\beta - x_0}(x - x_0). \end{aligned} \tag{4}$$

Combining (3) and (4) we have

$$f(x_0) + \frac{f(\beta) - f(x_0)}{\beta - x_0}(x - x_0) \leq f(x) \leq f(x_0) + \frac{f(x_0) - f(\alpha)}{x_0 - \alpha}(x - x_0)$$

and

$$\frac{f(\beta) - f(x_0)}{\beta - x_0}(x - x_0) \leq f(x) - f(x_0) \leq \frac{f(x_0) - f(\alpha)}{x_0 - \alpha}(x - x_0)$$

and

$$-M|x - x_0| \leq f(x) - f(x_0) \leq M|x - x_0|,$$

that is

$$|f(x) - f(x_0)| \leq M|x - x_0|$$

for

$$M = \max\left\{\frac{|f(\beta) - f(x_0)|}{\beta - x_0}, \frac{|f(x_0) - f(\alpha)|}{x_0 - \alpha}\right\}$$

and $\alpha < x < x_0$. For $x_0 < x < \beta$, we can similarly establish that

$$|f(x) - f(x_0)| \leq M|x - x_0|$$

for the same positive constant M. See Figure 3 for an illustration of the inequality

$$f(x_0) + \frac{f(x_0) - f(\alpha)}{x_0 - \alpha}(x - x_0) \leq f(x) \leq f(x_0) + \frac{f(\beta) - f(x_0)}{\beta - x_0}(x - x_0)$$

for $x_0 < x < \beta$.

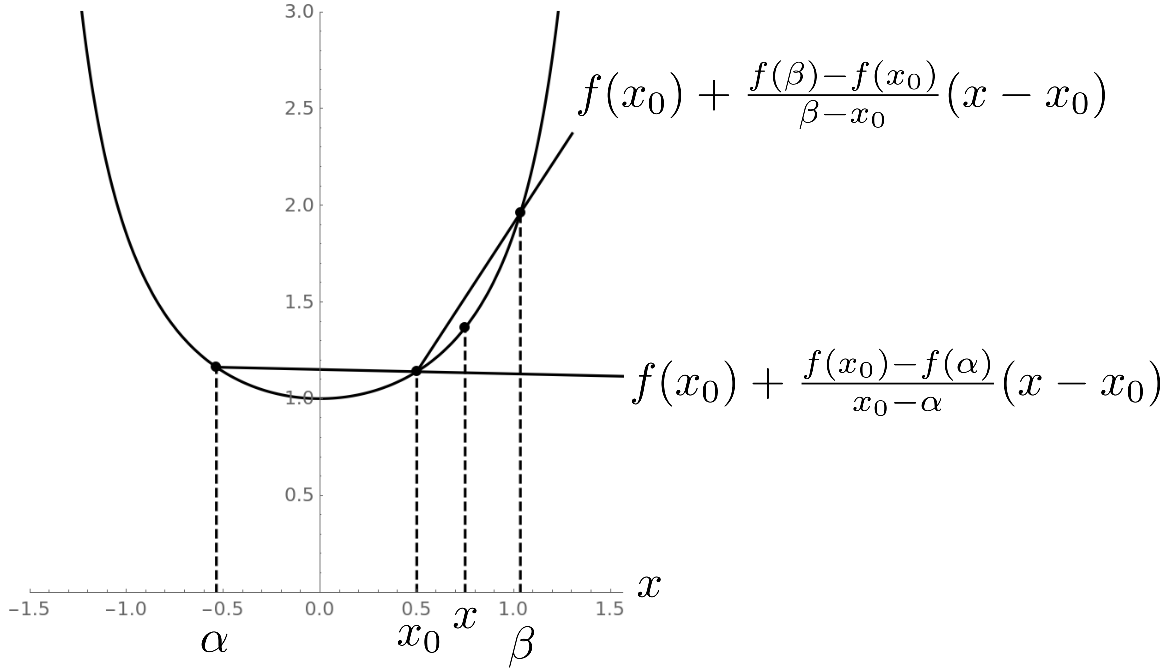

Figure 3: Bounds on $f(x)$ for $x_0 < x < \beta$.

Finally, letting $\epsilon > 0$ be arbitrary, we have that if

$$|x - x_0| < \frac{\epsilon}{M},$$

then for $\alpha \leq x \leq \beta$

$$|f(x) - f(x_0)| \leq M|x - x_0| < M\ \frac{\epsilon}{M} = \epsilon.$$

We conclude f is continuous at x_0.

Here the constant M does not depend on x but does depend on a, b, and x_0.

Problem 3 Review the definitions in Problem 8 (continuity) and Problem 9 (differentiability) of Assignment 1. Show that if a function $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x \in (a, b)$, then f is continuous at x.

Solution: Let $\epsilon > 0$. By differentiability there is some $\delta_1 > 0$ so that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon \qquad \text{whenever} \qquad 0 < |h| < \delta_1.$$

Notice that this inequality implies

$$|f(x+h) - f(x) - hf'(x)| < \epsilon|h|,$$

and if we relax the strict inequality, we can say

$$|f(x+h) - f(x) - hf'(x)| \leq \epsilon|h| \qquad \text{whenever} \qquad |h| < \delta_1.$$

In particular, if

$$|\xi - x| < \delta = \min\left\{\delta_1, \frac{1}{2}, \frac{\epsilon}{2(1 + |f'(x)|)}\right\},$$

and we take $h = \xi - h$ with $|\xi - h| < \delta$, then

$$\begin{aligned}
|f(\xi) - f(x)| &= |f(x+h) - f(x)| \\
&= |f(x+h) - f(x) - hf'(x) + hf'(x)| \\
&\leq |f(x+h) - f(x) - hf'(x)| + |hf'(x)| \\
&\leq \epsilon|h| + |h||f'(h)| \\
&< \epsilon\,\frac{1}{2} + \frac{\epsilon}{2(1 + |f'(x)|)}\;|f'(h)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This shows f is continuous at x.

Problem 4 Draw a picture of the **graph**

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in A\}$$

where $A = (0, 1) \times (0, 2)$ is an open rectangle and

$$u(x, y) = 1 + x^2.$$

Draw in red the curves

$$\{(1/2 + t, 1, u(1/2 + t, 1)) : 0 \leq t \leq 1/2\} \subset \mathcal{G}$$

and

$$\{(1/2, 1 + t, u(1/2, 1 + t)) \in \mathbb{R}^2 : 0 \leq t \leq 1/2\}.$$

Illustrate the (two) difference quotients

$$\frac{u(1/2 + h, 1) - u(1/2, 1)}{h} \qquad \text{and} \qquad \frac{u(1/2, 1 + h) - u(1/2, 1)}{h}$$

for u at $(1/2, 1)$ with $h = 1/4$.

Solution: Write

$$\Gamma_1 = \{(1/2 + t, 1, u(1/2 + t, 1)) : 0 \leq t \leq 1/2\}$$

and

$$\Gamma_2 = \{(1/2, 1 + t, u(1/2, 1 + t)) \in \mathbb{R}^2 : 0 \leq t \leq 1/2\}.$$

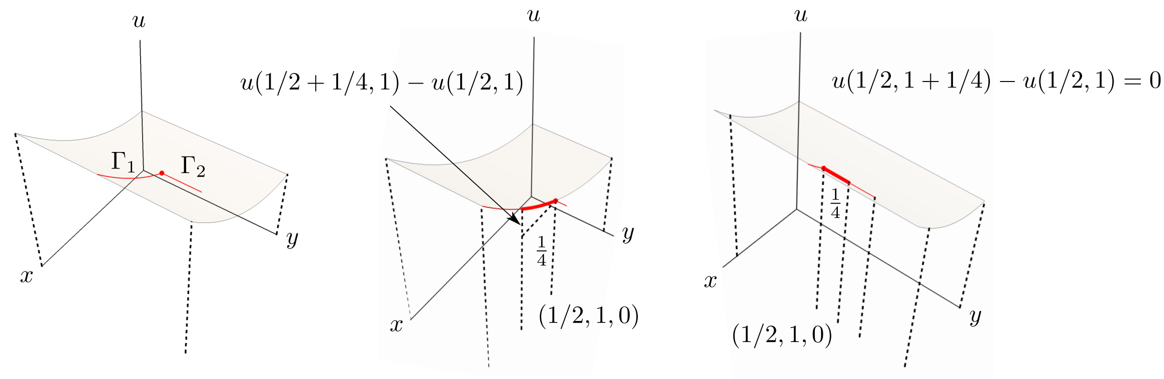

Figure 4: The graph $\mathcal{G}$ with the curve Γ_1 and Γ_2 (left). The x-difference quotent (middle). The y-difference quotient with zero increment in the codomain (right).

Note that Γ_2 is a horizontal segment and the codomain increment is $u(1/2, 1 + 1/2) - u(1/2, 1) = 0$.

Problem 5 Give an example to show that the existence of the partial derivatives

$$\frac{\partial u}{\partial x_j}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_j) - u(\mathbf{x})}{h}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and each $j = 1, \ldots, n$, i.e., partial differentiability, does not imply continuity (when $n > 1$).

Solution: Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ with values given by

$$u(x, y) = \begin{cases} \frac{(x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0). \end{cases}$$

It is clear that

$$\frac{\partial u}{\partial x}(x, y) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x, y)$$

exist when $(x, y) \neq (0, 0)$ since in this case the denominator $(x^2 + y^2)^2$ is nonzero, and the local expression for the function is entirely nonsingular. When either $x = 0$ or $y = 0$, there holds

$$u(x, y) = 1.$$

Consequently,

$$\frac{\partial u}{\partial x}(0, 0) = \lim_{v \to 0} \frac{u(v, 0) - u(0, 0)}{v} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(0, 0) = \lim_{v \to 0} \frac{u(0, v) - u(0, 0)}{v} = 0.$$

Thus, this function has well-defined partial derivatives at every point $(x, y) \in \mathbb{R}^2$.

On the other hand, if $x = y \neq 0$, then

$$u(x, y) = -4\frac{x^4}{(2x^2)^2} \equiv -1.$$

In particular,

$$\lim_{x \to 0} u(x, x) = -1 \neq u(0, 0) = 1.$$

Thus, $u \notin C^0(\mathbb{R}^2)$.

Problem 6 (increments and tolerances; Boas Problem 4.4.1) Consider $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^3}.$$

Note that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -\frac{3}{x^4}.$$

(a) Given $x > 0$ and a tolerance $\epsilon_1 > 0$, find a tolerance $\delta_1 > 0$ so that

$$0 < |h| < \delta_1 \qquad \text{implies} \qquad \left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| < \epsilon_1. \tag{5}$$

Hint: Assume $\delta_1 < \min\{x/2, \delta\}$ where δ is some other number. Use the estimate $\delta_1 < x/2$ to simplify the "extraneous" algebraic expression you obtain from simplifying the quantity to be estimated in (5). Then determine how small you need to make δ. Your answer should depend on ϵ_1 and x.

(b) Given $x > 0$ and a tolerance $\epsilon_0 > 0$, find a tolerance δ_0 for which

$$0 < |h| < \delta_0 \qquad \text{implies} \qquad \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| < \epsilon_0. \tag{6}$$

Hint: Use the same approach as in part **(a)**; but this one is easier.

(c) Say you can't pick the tolerance δ_0 in part **(b)**, but you are stuck with $|h| \le \delta_0 = 1$. What is the best tolerance ϵ_0 you can get in (6)?

Solution: Let's just start by having a look at the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left(\frac{1}{(x+h)^3} - \frac{1}{x^3} \right) = -\frac{h^2 + 3xh + 3x^2}{x^3(x+h)^3}. \tag{7}$$

Thus, if we take the difference we have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} &= \frac{3(x+h)^3 - xh^2 - 3x^2h - 3x^3}{x^4(x+h)^3} \\
&= \frac{h(6x^2 + 8xh)}{x^4(x+h)^3} \\
&= 2h\left(\frac{3}{x^2(x+h)^3} + \frac{4h}{x^3(x+h)^3} \right),
\end{aligned}$$

and for part **(a)** we want to take h small so this is smaller than some specified increment ϵ_1.

(a) The hint suggests taking $|h| < x/2$, and we can see immediately that this gives some relief from those denominators, since this means

$$x + h > \frac{x}{2} > 0 \qquad \text{and} \qquad 0 < \frac{1}{x+h} < \frac{2}{x}.$$

With this estimate we have

$$\left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| \leq 2|h| \left(\frac{24}{x^5} + \frac{32|h|}{x^6} \right) = \frac{16|h|}{x^6}(3x + 4|h|).$$

Again using $|h| < x/2$ we can simplify this to

$$\left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| \leq \frac{16(5)|h|}{x^5}.$$

Clearly, then if we have

$$|h| < \delta_1 = \min \left\{ \frac{x}{2}, \frac{x^5}{80}\ \epsilon_1 \right\}$$

then there will hold

$$\left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| < \epsilon_1.$$

(b) Here we are asked to estimate the simple difference $|f(x+h) - f(x)|$, so I can multiply the expression in (7) by h to get

$$\left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| \leq \frac{h^2 + 3x|h| + 3x^2}{x^6}\ (8|h|) \leq \frac{56|h|}{x^4}$$

as long as $|h| < x/2$. Thus, taking

$$|h| < \delta_0 = \min \left\{ \frac{x}{2}, \frac{x^4}{56}\ \epsilon_1 \right\}$$

gives the desired conclusion. The restriction $|h| > 0$ is unnecessary in this case.

(c) This is rather harder if we really want the best possible tolerance. The point is that the expression

$$\alpha(h) = \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| = \frac{|h|}{x^3} \left| \frac{h^2 + 3xh + 3x^2}{(x+h)^3} \right|$$

should have a maximum value for some h with $|h| \le 1$. This maximum value is the best tolerance for which we can hope if we only know $|h| \le 1$. It's just a calculus problem of course. For $h \neq 0$ and $h \neq -x$ we have

$$x^3\alpha'(h) = \frac{h(2h+3x)+h^2+3xh+3x^2}{(x+h)^3} - \frac{3h(h^2+3xh+3x^2)}{(x+h)^4} = \frac{3x^2}{(x+h)^4}.$$

Since this derivative doesn't vanish, we can consider (mostly) the boundary values $h = \pm 1$. We find

$$\alpha(-1) = -\frac{3x^2-3x+1}{x^3(x-1)^3} \qquad \text{and} \qquad \alpha(1) = \frac{3x^2+3x+1}{x^3(x+1)^3}.$$

Notice that if x gets close to $x = 1$ with $x < 1$, then the expression for $\alpha(-1)$ tends to $+\infty$. This looks like trouble for a maximum value.

Returning to the original expression when $x = 1$, we have

$$\alpha(h) = \left| \frac{1}{(1+h)^3} - 1 \right| \ge \frac{1}{(1+h)^3} - 1$$

and

$$\lim_{h \searrow -1} \frac{1}{(1+h)^3} = +\infty.$$

Thus, if $x = 1$, all bets are off. That is to say one can't get any finite tolerance whatsoever in (6). This same problem persists for any x with $0 < x \le 1$, because then the original expression has

$$\lim_{h \searrow -x} \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| = +\infty.$$

If on the other hand, $x > 1$, then we should be able to get a tolerance. In that case,

$$\alpha(1) - \alpha(-1) = \frac{3x^2+3x+1}{x^3(x+1)^3} + \frac{3x^2-3x+1}{x^3(x-1)^3} = 2\frac{3x^2+1}{(x^2-1)^3} > 0,$$

so

$$\alpha(1) = \frac{3x^2+3x+1}{x^3(x+1)^3} > 0$$

is the maximum value, and this is the tolerance we can expect: If $x > 1$, then

$$\left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| < \frac{3x^2 + 3x + 1}{x^3(x+1)^3}$$

if $|h| < 1$. If you can have $h = 1$ as I've written, then you can have equality

$$\left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| \leq \frac{3x^2 + 3x + 1}{x^3(x+1)^3},$$

so if you really want an estimate like (6) with strict inequality, you can take any number ϵ_0 with

$$\epsilon_0 > \frac{3x^2 + 3x + 1}{x^3(x+1)^3}.$$

Technically, in this case there is no "best" tolerance in (6). On the other hand (6) itself contains the hypothesis $|h| < \delta_0$, so probably what I meant to say in part **(c)** is that you are stuck with $|h| < \delta_0 = 1$ instead of $|h| \leq \delta_0 = 1$ (which is what I actually typed). In any case, the solution above hopefully makes the overall situation clear and might even be correct.

Problem 7 A function $f : \mathbb{C} \to \mathbb{C}$ with real and imaginary parts expressed as functions $u, v \in C^2(\mathbb{R}^2)$ so that $f(x + iy) = u(x, y) + v(x, y)\, i$ is said to be **complex differentiable** at $z = x + iy \in \mathbb{C}$ if

$$f'(z) = \lim_{h \to 0+0i} \frac{f(z+h) - f(z)}{h}$$

exists. Show that if f is complex differentiable at every $z = x + iy \in \mathbb{C}$, then the functions u and v satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Hint: Take the limit first in the special case where $h = \epsilon \in \mathbb{R}$. Then take the limit in the special case where $h = \epsilon\, i$. In each case you should get answers involving first order partial derivatives of u and v. Because the answers you get look different, but according to the definition of what it means to be complex differentiable must be the same, you should be able to obtain some interesting relations among the first partial derivatives.

Solution: First we take the limit of the complex difference quotient with increment $h + ik = h \in \mathbb{R}$ keeping in mind the one-to-one correspondence $\psi : \mathbb{C} \to \mathbb{R}^2$ with $\psi(x + iy) = (x, y)$:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h}.$$

After algebraic rearrangement and taking the real limits, we find

$$f'(z) = u_x(x, y) + iv_x(x, y). \tag{8}$$

Next, we take the same limit with a purely complex increment $h + ik = ik$:

$$\begin{aligned} f'(z) &= \lim_{k \to 0} \frac{u(x, y+k) + iv(x, y+k) - u(x, y+k) - iv(x, y+k)}{ik} \\ &= \frac{1}{i}[u_y(x, y) + iv_y(x, y)] \\ &= v_y(x, y) - iu_y(x, y) \end{aligned}$$

because $1/i = i/(i^2) = -i$. Equating the two expressions for $f'(z)$ we see

$$u_x + iv_x = v_y - iu_y.$$

Further equating the real and imaginary parts gives the Cauchy-Riemann equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$$

Since $u, v \in C^2(\mathbb{R}^2)$, there is no problem taking second derivatives:

$$\begin{cases} u_{xx} = v_{xy} \\ u_{xy} = -v_{xx} \end{cases} \qquad \text{and} \qquad \begin{cases} u_{xy} = v_{yy} \\ u_{yy} = -v_{xy}. \end{cases}$$

The first and last equation give

$$u_{xx} = v_{xy} = -u_{yy} \qquad \text{or} \qquad \Delta u = 0.$$

Similarly,

$$\Delta v = v_{xx} + v_{yy} = -u_{xy} + u_{xy} = 0.$$

Problem 8 (Exercise 1.26 in my notes) Recall the following definition:

Definition 1 (differentiability for a function of several variables) Given an open set $U \subset \mathbb{R}^n$ and $u : U \to \mathbb{R}$, we say u is **differentiable** at $\mathbf{p} \in U$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p}) = \circ(\|\mathbf{x} - \mathbf{p}\|)$$

as $\mathbf{x} \to \mathbf{p}$. The notation on the right here is read "little-o of $\|\mathbf{x} - \mathbf{p}\|$." It means the **limit of the quotient** of $u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})$ and $\|\mathbf{x} - \mathbf{p}\|$ is zero as $\mathbf{x}$ tends to $\mathbf{p}$, or more properly for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{p}\| < \delta \qquad \text{implies} \qquad \left| \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|} \right| < \epsilon.$$

Show differentiability implies partial differentiability. Hint: One way that $\mathbf{x}$ can limit to $\mathbf{p} \in U$ is in the form $\mathbf{x} + h\mathbf{e}_j$ as h tends to zero.

Solution: First let $L = L(\mathbf{e}_1)$. (This is called an "abuse of notation" because I'm using L to mean two different things here. On the one hand, $L : \mathbb{R}^n \to \mathbb{R}$ is some linear function. On the other hand, I've decided to let the symbol L represent one particular value $L(\mathbf{e}_1) \in \mathbb{R}$. We'll have to be a little careful to keep track of the meaning as we read the solution. A safer and perhaps better way would be to introduce a different symbol and write something like $\ell = L(\mathbf{e}_1) \in \mathbb{R}$ or $M = L(\mathbf{e}_1) \in \mathbb{R}$. What I'm doing can be dangerous...real "living on the edge" for a mathematician.)

For any $\epsilon > 0$ there is some $\delta > 0$ so that

$$0 < \|\mathbf{x} - \mathbf{p}\| < \delta \qquad \text{implies} \qquad \left| \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|} \right| < \epsilon.$$

In particular, if we set $\mathbf{x} = \mathbf{p} + h\mathbf{e}_1$ and $0 < |h| < \delta$, then we should have

$$\left| \frac{u(\mathbf{p} + h\mathbf{e}_1) - u(\mathbf{p}) - L(h\mathbf{e})}{\|h\mathbf{e}_1\|} \right| < \epsilon.$$

Since L is a linear function $L(h\mathbf{e}_1) = hL(\mathbf{e}_1) = hL$. Also, $\|h\mathbf{e}_1\| = h$, so

$$\left| \frac{u(\mathbf{p} + h\mathbf{e}_1) - u(\mathbf{p})}{h} - L \right| < \epsilon$$

whenever $0 < |h| < \delta$. This is exactly what it means for the limit

$$\frac{\partial u}{\partial x_1}(\mathbf{p}) = \lim_{h \to 0} \frac{u(\mathbf{p} + h\mathbf{e}_1) - u(\mathbf{p})}{h}$$

to exist.

There is nothing special about $\mathbf{e}_1$ in this argument. We could just as easily have used $\mathbf{e}_j$ for any $j = 1, 2, \ldots, n$ in place of $\mathbf{e}_1$. Thus, if u is differentiable at $\mathbf{p} \in U$, then

$$u_{x_j}(\mathbf{p}) = \frac{\partial u}{\partial x_j}(\mathbf{p})$$

exists for every $j = 1, 2, \ldots, n$. Thus, u is partially differentiable at $\mathbf{p}$.

Problem 9 (partial derivatives in Problem 7 above) In Problem 7 above you should have noticed that the partial derivatives of u are given by

$$\frac{\partial u}{\partial x_j}(\mathbf{p}) = L(\mathbf{e}_j)$$

where $L = du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$ is the **differential map**. What does this tell you about the linear function $L = du_{\mathbf{p}}$? Hint: What do you know about a real valued linear map $L : \mathbb{R}^n \to \mathbb{R}$ (from linear algebra)?

Solution: A linear map $L : \mathbb{R}^n \to \mathbb{R}$ is given by a dot product

$$L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{w}. \tag{9}$$

This is a special case of what is called the Riesz representation theorem. Furthermore, hopefully you remember from linear algebra that the vector $\mathbf{w}$ has components/entries given by the value of L on the standard unit basis vectors. That is in this case

$$\mathbf{w} = (L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)).$$

Since

$$L(\mathbf{e}_j) = \frac{\partial u}{\partial x_j}$$

we see the vector $\mathbf{w}$ is the vector containing the first partials as entries, that is the gradient vector Du. We conclude

$$L(\mathbf{v}) = Du \cdot \mathbf{v}.$$

In words, *the linear map L is the dot product with the gradient.*

Problem 10 Note the following:

(i) In Problem 5 above you showed partial differentiability does not imply continuity.

(ii) In Problem 8 above you showed differentiability implies partial differentiability.

Can you show differentiability implies continuity?

Solution: We should be able to adapt the solution of Problem 3 above to higher dimensions. We need one additional estimate: Recall from Problem 9 that

$$du_{\mathbf{p}}(\mathbf{v}) = Du(\mathbf{p}) \cdot \mathbf{v}.$$

The Cauchy-Schwarz inequality than implies

$$|du_{\mathbf{p}}(\mathbf{v})| \le |Du(\mathbf{p})||\mathbf{v}| \tag{10}$$

which is an estimate that holds for all $\mathbf{v} \in \mathbb{R}^n$.

Let $\epsilon > 0$. By differentiability there is some $\delta_1 > 0$ so that

$$\left| \frac{u(\mathbf{x}) - u(\mathbf{p})}{|\mathbf{x} - \mathbf{p}|} - \frac{du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})}{|\mathbf{x} - \mathbf{p}|} \right| < \epsilon \qquad \text{whenever} \qquad 0 < |\mathbf{x} - \mathbf{p}| < \delta_1.$$

Notice this inequality implies

$$|u(\mathbf{x}) - u(\mathbf{p}) - du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| < \epsilon |\mathbf{x} - \mathbf{p}|,$$

and if we relax the strict inequality, we can say

$$|U(\mathbf{x}) - u(\mathbf{p}) - du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| \le \epsilon |\mathbf{x} - \mathbf{p}| \qquad \text{whenever} \qquad |\mathbf{x} - \mathbf{p}| < \delta_1.$$

In particular, if

$$|\mathbf{x} - \mathbf{p}| < \delta = \min \left\{ \delta_1, \frac{1}{2}, \frac{\epsilon}{2(1 + |Du(\mathbf{p})|)} \right\},$$

then

$$\begin{aligned}
|u(\mathbf{x}) - u(\mathbf{p})| &= |u(\mathbf{x}) - u(\mathbf{p}) - du_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) + du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| \\
&\le |u(\mathbf{x}) - u(\mathbf{p}) - du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| + |du_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| \\
&\le \epsilon |\mathbf{x} - \mathbf{p}| + |Du(\mathbf{p})| \; |\mathbf{x} - \mathbf{p}| \\
&< \epsilon \, \frac{1}{2} + \frac{\epsilon}{2(1 + |Du(\mathbf{p})|)} \; |Du(\mathbf{p})| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This shows u is continuous at $\mathbf{p}$.