Assignment 2: Partial derivatives some solutions

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February 7, 2023

Problem 1 Draw the set

$$\{\mathbf{x} + h\mathbf{e}_j : |h| < \delta\} \subset \mathbb{R}^n$$

when

(a) $n = 2$, $\mathbf{x} = (2, 1)$ and $\delta = 1/3$.

(b) $n = 3$, $\mathbf{x} = (1, 0, 1/2)$ and $\delta = 1/4$.

Partial Solution: In general these are open line segments.

(a) If $j = 1$, then the drawing should look like this:

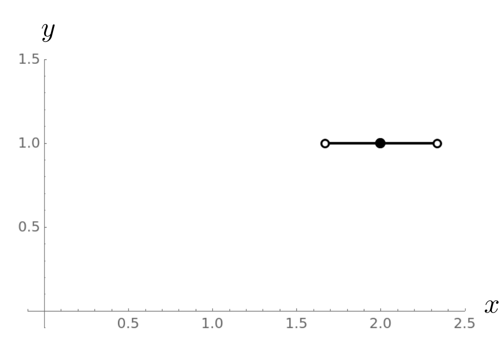

Figure 1: A drawing of the set $\{\mathbf{x} + h\mathbf{e}_1 : |h| < 1/3\} \subset \mathbb{R}^2$.

Problem 2 (convexity) Recall the following definition: A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \tag{1}$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \leq t \leq 1$.

Show that if f is convex, then

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{2}$$

Hint: Assume there is some $\epsilon > 0$ and a sequence of points $x_1, x_2, x_3, \ldots$ with $x_j \nearrow x_0$ and $f(x_j) \leq f(x_0) - \epsilon$. If you can get a contradiction out of this, it means

$$\lim_{x \nearrow x_0} f(x) \geq f(x_0).$$

What does (2) tell you about convex functions?

Problem 3 Draw a picture of the **graph**

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in A\}$$

where $A = (0, 1) \times (0, 2)$ is an open rectangle and

$$u(x, y) = 1 + x^2.$$

Draw in red the curves

$$\{(1/2 + t, 1, u(1/2 + t, 1)) : 0 \leq t \leq 1/2\} \subset \mathcal{G}$$

and

$$\{(1/2 + t, u(1/2 + t, 1)) \in \mathbb{R}^2 : 0 \leq t \leq 1/2\}.$$

Illustrate the (two) difference quotients

$$\frac{u(1/2 + h, 1) - u(1/2, 1)}{h} \qquad \text{and} \qquad \frac{u(1/2, 1 + h) - u(1/2, 1)}{h}$$

for u at $(1/2, 1)$ with $h = 1/4$.

Problem 4 Here is a definition of what it means for a subset of $\mathbb{R}^n$ to be open: A set $U \subset \mathbb{R}^n$ is **open** if for each

$$\mathbf{p} = \sum_{j=1}^{n} p_j \mathbf{e}_j \in U,$$

there is some $\delta > 0$ for which

$$Q_\delta(\mathbf{p}) = \left\{ \mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j : |x_j - p_j| < \frac{\delta}{2} \right\} \subset U. \tag{3}$$

The set $Q_\delta(\mathbf{p})$ is called the **open cube of side length δ centered at p**.

Show that for $\delta > 0$ and $\mathbf{p} \in \mathbb{R}^n$, the cube $Q_\delta(\mathbf{p})$ is open.

Note: It has come to my attention that some of you have difficulty reading the problem above. I did not intend for you to have that difficulty, but in retrospect I can understand to some extent why you do have difficulty. For that reason, let me offer the following "reading hints" for this problem. If you have no difficulty reading the problem, you can ignore this and just skip to the solution below, though I do make some reference to this discussion there. If you have any difficulty understanding the statement of this problem, I hope this note will remedy that situation though it will require (as with most mathematical topics) some patience and attention to detail.

First note the large scale structure of the problem:

1. First there is a definition (of what it means to be an **open set**). This starts with the first colon and ends after the displayed expression (3).

2. After that is an explanatory comment, noting that some notation (for an object called a **cube**) has been introduced within the definition. In effect, I have defined a cube within the definition of an open set. That is, there is a definition within the definition. This may have potentially been more clearly expressed if I defined what it means for a set to be a cube first, before I gave the definition of an open set. (It may be helpful for you to rewrite the problem giving an exposition in this way: "First I'm going to define what it means for a set to be a cube with side length δ and center $\mathbf{p}$. ...Now, I'm going to define what it means for a set to be open...." I'll leave that to you.)

3. Finally, there is the statement of the problem starting with the word "Show."

When you encounter a problem like this, where a definition (or two) is given first, then (very likely) you will need to "internalize" the definition before you will be able to solve the problem. That is, you need to understand what is being said. So be patient and think about the definitions first. You can start with the notion of a cube. It may help to draw what a cube looks like in dimensions $n = 1$, $n = 2$, and $n = 3$, where the pictures are relatively easy to draw.

Once you understand, i.e., have internalized, what a cube is and how it works, then you can move on to the definition of an open set. Again, drawing a picture or two may be helpful. For example, $B_1(\mathbf{0}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open subset of $\mathbb{R}^2$. You can draw this unit disk with center at the origin $\mathbf{0} = (0, 0)$ and consider various points $\mathbf{p} = (p_1, p_2)$ within it. Each point $\mathbf{p}$ you consider should have associated with it a cube $Q_\delta(\mathbf{p})$ of some side length δ satisfing $Q_\delta(\mathbf{p}) \subset B_1(\mathbf{0})$. You should note that the value of δ will depend on the particular point $\mathbf{p}$ you have chosen. (For points $\mathbf{p}$ closer to the circle $\partial B_1(\mathbf{0})$ you'll need smaller side lengths.)

When you read my definition of an open set, it may (hopefully) seem pretty straightforward until you read the display (3) the complexity of which you may find alarming. Leaving out some notation, it says a set U is open **if** whenever you take a point $\mathbf{p} \in U$, you can find some positive number δ so that (3) holds. If this was your experience, then let me suggest to you the following psychological "trick." Scan through (3) and notice first the fact that there is a period at the end of (3). Thus, you can say to yourself the following:

> *Whatever is in* (3) *looks complicated and like it will take some time to understand, but whatever it is, when I understand what is being said there, I will be done with this definition and I'll understand the entire thing.*

After telling yourself this, hopefully, you'll conclude it is worth it to be patient and figure out whatever it is that is being said in (3). You might also go ahead and scan the next line from which you may pick up that fact that a big part of (3) is nothing more than the definition of a new kind of set. In fact, that is most of what looks complicated. From the point of view of the definition of an open set, all (3) is saying is that the cube $Q_\delta(\mathbf{p})$ is a subset of U. What is written there just looks big and intimidating because it also contains the definition of a cube.

As a final general point about reading such a problem (and especially this problem): When you finish the "definition" part of the statement and begin the "problem" part of the statement, you will need to be able to "let go" of the notation used in the definition. That is to say, a definition is intended to communicate a certain concept. It is assumed the concept will be internalized, and when the concept is internalized the notation becomes incidental. I used symbols like δ and $\mathbf{p}$ to give the definition of a cube with center $\mathbf{p}$ and side length δ, but when I am done I "know" what a cube is, and I can probably express my understanding in a way that "frees up" the symbols δ and $\mathbf{p}$ for other uses. Let me give it a try:

> *A* **cube** *is a set in which the coordinates of each point satisfy the condition that the absolute value of the difference of each coordinate and the corresponding coordinate of some fixed point, called the center, is strictly less than some positive tolerance. Twice the tolerance is called the side length of the cube.*

Of course, my definition here is (perhaps) somewhat more difficult to understand than the definition given in the problem, but it shows that I understand the definition. Most importantly, it shows that the symbols δ and $\mathbf{p}$ are not essential parts of the definition. I did not use those symbols. Thus, those symbols are "freed up" in my mind to be used for other purposes. What I am describing here is important for this problem. When I start the statement of the "problem" part with the word "Show," the symbols δ and $\mathbf{p}$ are about to be used in a context fundamentally distinct from their use in the definition immediately preceeding. So you need to "clean the slate" in your mind. When you consider $Q_\delta(\mathbf{p})$ in the "problem" part, the symbols δ and $\mathbf{p}$ have little or nothing to do with the symbols δ and $\mathbf{p}$ used in the predeeding definition. To state the problem I could have said

> *Show that for $r > 0$ and $\mathbf{a} \in \mathbb{R}^n$, the cube $Q_r(\mathbf{a})$ is open.*

This might have made reading the problem easier for you but, from the point of view of mathematical exposition, it is irritating, and you won't find this sort of thing in the literature because it just uses too many unnecessary symbols. So I suggest you just take the time to learn how to read such things. At least that's what I'm hoping you might do.

Solution: We apply the definition (of what it means for a set to be open), which happens to include the definition of what it means for a set to be a cube, to the particular set $U = Q_\delta(\mathbf{p})$ (which is a cube). In order to apply the definition, we must begin with an arbitrary point in $U = Q_\delta(\mathbf{p})$. Let us call this point $\mathbf{q}$. Thus, we begin with $\mathbf{q} \in Q_\delta(\mathbf{p})$. Naturally, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ has some components $q_1, q_2, \ldots, q_n \in \mathbb{R}$. Another way to say express this obvious fact is to write

$$\mathbf{q} = \sum_{j=1}^{n} q_j \mathbf{e}_j.$$

In such an expression, the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit basis vectors already featured in Problem 1 above and discussed in Chapter 1 (section 1.1 page 15 under the heading "some notation from linear algebra"). So then, we begin with an arbitrary point $\mathbf{q} \in Q_\delta(\mathbf{p})$. By the definition of the cube $Q_\delta(\mathbf{p})$, we know

$$|q_j - p_j| < \frac{\delta}{2} \qquad \text{for} \qquad j = 1, 2, \ldots, n.$$

This means, in particular, that we have on hand n positive numbers

$$\frac{\delta}{2} - |q_j - p_j| > 0.$$

We need to find a positive side length, i.e., number/tolerance, so that the cube of that side length centered at $\mathbf{q}$ is totally inside $Q_\delta(\mathbf{p})$. We should **not** use the symbol δ for this side length. Though this is the symbol used for the side length of the cube in the definition, we have now moved to the "problem" portion, and the symbols δ and $\mathbf{p}$ are already in use in a different context. So I need to use different symbols. I've already started doing that when I chose the point $\mathbf{q}$. So let me call the side length ϵ. I'm looking for some $\epsilon > 0$ for which $Q_\epsilon(\mathbf{q}) \subset U = Q_\delta(\mathbf{p})$. Here is my choice:

$$\epsilon = \min\{\delta - 2|q_j - p_j| : j = 1, 2, \ldots, n\}. \tag{4}$$

This same number can be expressed as

$$\epsilon = \min_{1 \le j \le n} (\delta - 2|q_j - p_j|)$$

or

$$\epsilon = \delta - 2 \max_{1 \le j \le n} |q_j - p_j|.$$

These are all the same number. The first two expressions are nice because I know each of the numbers $\delta - 2|q_j - p_j|$ is positive, and so the minimum of a finite set of positive numbers will also be positive. That is, $\epsilon > 0$.

Finally, I want to check that my choice of side length "works." That is, I need to show $Q_\epsilon(\mathbf{q}) \subset U = Q_\delta(\mathbf{p})$. For this, I should take an arbitrary point in $Q_\epsilon(\mathbf{q})$ and show that point must also be in $Q_\delta(\mathbf{p})$. Again, I need another symbol, and $\mathbf{p}$ and $\mathbf{q}$ are already in use. I will use $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ which is free at the moment. If $\mathbf{x} \in Q_\epsilon(\mathbf{q})$, then I know

$$|x_j - q_j| < \frac{\epsilon}{2} \qquad \text{for} \qquad j = 1, 2, \ldots, n.$$

By the traingle inequality I get

$$\begin{aligned}|x_j - p_j| &\leq |x_j - q_j| + |q_j - p_j| \\ &< \frac{\epsilon}{2} + |q_j - p_j| \\ &\leq \frac{\delta}{2} - |q_j - p_j| + |q_j - p_j| \qquad (5) \\ &= \frac{\delta}{2}.\end{aligned}$$

The second to last estimate (5) follows from the definition of ϵ in (4). Finally, I conclude $|x_j - p_j| < \delta/2$ for $j = 1, 2, \ldots, n$, and this is what it means for $\mathbf{x}$ to be in $Q_\delta(\mathbf{p})$. I have shown $Q_\epsilon(\mathbf{q}) \subset Q_\delta(\mathbf{p})$. Thus, I have shown $Q_\delta(\mathbf{p})$ is open. $\square$. (Sometimes it's nice to put a box at the end of an argument to let the sleepy reader know there is nothing more on this topic forthcoming.[1])

[1] The mathematician Carl Gauss popularized this idea by writing QED at the end of his arguments. Roughly speaking this is an acronym for "What was to be shown," i.e., "I have proved that which was to be shown." In Latin it's "quod erat demonstrandum." Apparently, Gauss anticipated a lot of sleepy readers.

Problem 5 (increments and tolerances; Boas Problem 4.4.1) Consider $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^3}.$$

Note that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -\frac{3}{x^4}.$$

(a) Given $x > 0$ and a tolerance $\epsilon_1 > 0$, find a tolerance $\delta_1 > 0$ so that

$$0 < |h| < \delta_1 \qquad \text{implies} \qquad \left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| < \epsilon_1. \tag{6}$$

Hint: Assume $\delta_1 < \min\{x/2, \delta\}$ where δ is some other number. Use the estimate $\delta_1 < x/2$ to simplify the "extraneous" algebraic expression you obtain from simplifying the quantity to be estimated in (6). Then determine how small you need to make δ. Your answer should depend on ϵ_1 and x.

(b) Given $x > 0$ and a tolerance $\epsilon_0 > 0$, find a tolerance δ_0 for which

$$0 < |h| < \delta_0 \qquad \text{implies} \qquad \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| < \epsilon_0. \tag{7}$$

Hint: Use the same approach as in part **(a)**; but this one is easier.

(c) Say you can't pick the tolerance δ_0 in part **(b)**, but you are stuck with $|h| \le \delta_0 = 1$. What is the best tolerance ϵ_0 you can get in (7)?

Solution:

(a) Let

$$\delta_1 = \min\left\{ \frac{x}{2}, \frac{x^5}{96}\epsilon_1 \right\}. \tag{8}$$

Then if $0 < |h| < \delta_1$ we have

$$\begin{aligned}
\left|\frac{f(x+h)-f(x)}{h} + \frac{3}{x^4}\right| &= \left|\frac{1}{hx^3(x+h)^3}[x^3-(x+h)^3] + \frac{3}{x^4}\right| \\
&= \left|\frac{1}{hx^3(x+h)^3}(-h)[x^2+x(x+h)+(x+h)^2] + \frac{3}{x^4}\right| \\
&= \left|-\frac{x^2+x(x+h)+(x+h)^2}{x^3(x+h)^3} + \frac{3}{x^4}\right| \\
&= \left|\frac{3}{x^4} - \frac{x^2+x(x+h)+(x+h)^2}{x^3(x+h)^3}\right| \\
&= \left|\frac{3(x+h)^3 - x[x^2+x(x+h)+(x+h)^2]}{x^4(x+h)^3}\right| \\
&= \left|\frac{3(x^3+3x^2h+3xh^2+h^3) - x[3x^2+3xh+h^2]}{x^4(x+h)^3}\right| \\
&= \frac{|h|}{x^4|x+h|^3}|6x^2+8xh+3h^2| \\
&\le \frac{|h|}{x^4(x/2)^3}|6x^2+8xh+3h^2| \qquad (9) \\
&\le \frac{|h|}{x^4(x/2)^3}\ 12x^2 \qquad (10) \\
&= \frac{96|h|}{x^5} \qquad (11) \\
&< \epsilon_1.
\end{aligned}$$

The estimate (9) follows because $|x+h| > x/2$. This follows, in turn, from the triangle inequality becuase $x = |x| = |x+h-h| \le |x+h| + |h| < |x+h| + x/2$. The estimate (10) also uses (only) $|h| < x/2$ and the triangle inequality:

$$|6x^2+8xh+3h^2| \le 6x^2 + 8x|h| + 3h^2 < 6x^2 + 4x^2 + \frac{3x^2}{4} < 10x^2 + 2x^2.$$

The expression in (11) explains the choice of δ_1 in (8).

(b) We can proceed in much the same way: Let

$$\delta_0 = \min\left\{\frac{x}{2}, \frac{x^4}{48}\,\epsilon_0\right\}.$$

Then $0 < |h| < \delta_0$ implies

$$\begin{aligned}
\left|\frac{1}{(x+h)^3} - \frac{1}{x^3}\right| &= \left|\frac{1}{(x+h)^3} - \frac{1}{x^3}\right| \\
&= \left|-h\ \frac{x^2 + x(x+h) + (x+h)^2}{x^3(x+h)^3}\right| \\
&= |h|\left|\frac{x^2 + x(x+h) + (x+h)^2}{x^3(x+h)^3}\right| \\
&= |h|\left|\frac{x^2 + x(x+h) + (x+h)^2}{x^3(x+h)^3}\right| \\
&= |h|\frac{|3x^2 + 3xh + h^2|}{x^3|x+h|^3} \\
&< |h|\frac{3x^2 + 3x^2/2 + 3x^2/4}{x^3|x+h|^3} \\
&< |h|\frac{3x^2 + 2x^2 + x^2}{x^3(x+h)^3} \\
&< \frac{48|h|}{x^4} \\
&< \epsilon_0.
\end{aligned}$$

(c) This part is potentially much more difficult, specifically because I asked for the "best" estimate. Immediately, this means all the simplification of assuming $|h| < x/2$ is unavailable. This suggests, furthermore, that to do this correctly one is going to have to consider some function of the tolerance h and execute some kind of maximization. Fortunately, the answer is geometrically obvious, so we can use this to guide the estimates and analysis if necessary. One can see from the graph of $f(x) = 1/x^3$ shown in Figure 2 that the prospects for bounding the increment $|f(x+h) - f(x)|$ for $0 < |h| < 1$ are very different when $x \le 1$ and when $x > 1$. If $x \le 1$ as indicated on the left in Figure 2, then by taking $-1 < h < 0$ with h very close to $-x$, we can obtain arbitrarily large values for the increment $|f(x+h) - f(x)|$. Therefore, **no finite tolerance** ϵ_0 **can be attained** as the problem suggests under the assumption $|h| < 1$ when $0 < x \le 1$. If, on the other hand, $x > 1$ as illustrated on the right in Figure 2, then we have $0 < x - 1 < x + h < x + 1$ if $|h| < 1$, and an upper bound for the maximum increment is clearly seen to be given by $|f(x-1) - f(x)|$.

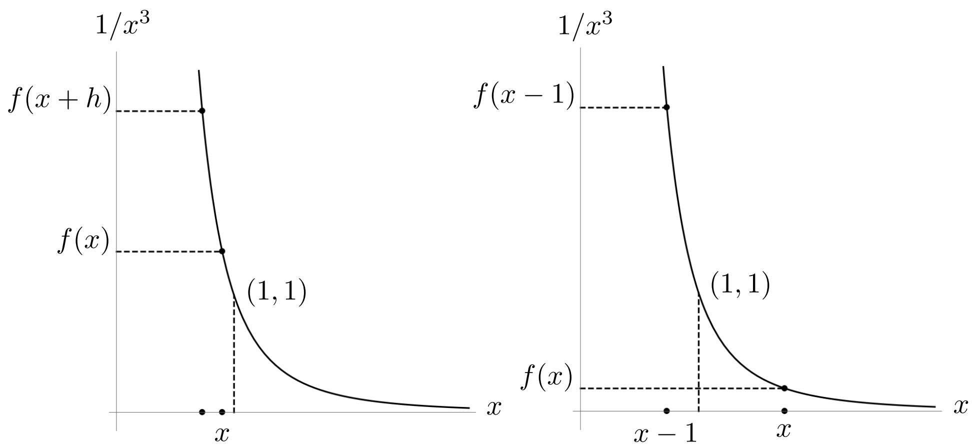

Figure 2: The graph of $1/x^3$.

To analytically verify these assertions, we proceed as follows: We will first show the difference $|f(x+h)-f(x)|$ can always be made arbitrarily large when $x \leq 1$. Let $N > 1$ be arbitrary. Note that we can take some $M > 0$ satisfying

$$M^3 > \max\left\{\frac{1}{x}, N + \frac{1}{x^3}\right\}.$$

It follows that $-1 < 1/M - x < 0$, so $h = 1/M - x$ is an admissible increment satisfying $|h| < 1$. The associated increment in f, however, satisfies

$$|f(1/M) - f(x)| \geq M^3 - \frac{1}{x^3} > N.$$

This shows it is not possible for find any finite tolerance ϵ_0 as suggested when $x \leq 1$.

When $x > 1$, on the other hand, we claim

$$\left|\frac{1}{(x+h)^3} - \frac{1}{x^3}\right| \leq f(x-1) - f(x) = \frac{1}{(x-1)^3} - \frac{1}{x^3} = \frac{3x^2 - 3x + 1}{x^3(x-1)^3}.$$

If this claim is correct, then we have the desired tolerance

$$\epsilon_0 = f(x-1) - f(x)$$

since clearly

$$\epsilon_0 = \lim_{h \searrow -1} [f(x+h) - f(x)].$$

To see the claim, we consider the function $\phi : (-1, 1) \to \mathbb{R}$ given by

$$\phi(h) = f(x+h) - f(x).$$

Note that ϕ extends continuously to $-1 \leq h \leq 1$ and satisfies

$$\frac{d\phi}{dh} = f'(x+h) = -\frac{1}{(x+h)^3} < 0.$$

Thus ϕ is a decreasing function with a unique zero at $h = 0$ as indicated on the left in Figure 3.

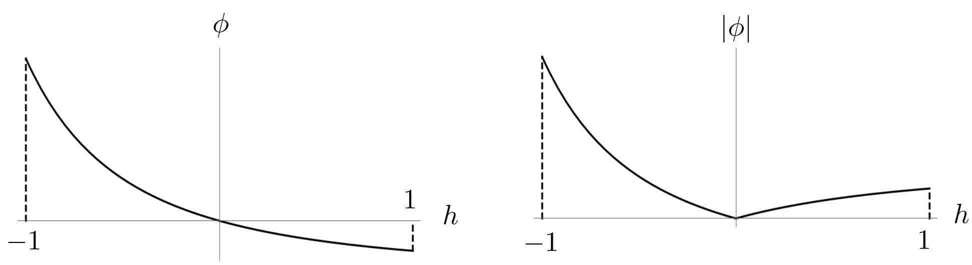

Figure 3: The graph of the increment $\phi = \phi(h)$.

This means $|\phi(h)|$ has precisely two local max values given by $|\phi(-1)|$ and $|\phi(1)|$. It remains to show $\epsilon_0 = \phi(-1) > -\phi(1)$. To see this, observe

$$\begin{aligned}
\phi(-1) + \phi(1) &= f(x-1) - f(x) + f(x+1) - f(x) \\
&= \frac{1}{(x-1)^3} - \frac{1}{x^3} - \left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right) \\
&= \left(\frac{1}{x-1} - \frac{1}{x}\right)\left(\frac{1}{(x-1)^2} + \frac{1}{x(x-1)} + \frac{1}{x^2}\right) \\
&\qquad - \left(\frac{1}{x} - \frac{1}{x+1}\right)\left(\frac{1}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}\right) \\
&= \frac{1}{x(x-1)}\left(\frac{1}{(x-1)^2} + \frac{1}{x(x-1)} + \frac{1}{x^2}\right) \\
&\qquad - \frac{1}{x(x+1)}\left(\frac{1}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}\right) \\
&= \frac{1}{x}\left[\frac{1}{x-1}\left(\frac{1}{(x-1)^2} + \frac{1}{x(x-1)} + \frac{1}{x^2}\right)\right. \\
&\qquad \left. -\frac{1}{x+1}\left(\frac{1}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}\right)\right].
\end{aligned}$$

Now notice first that

$$\frac{1}{x-1} > \frac{1}{x} > \frac{1}{x+1}.$$

It follows also that

$$\frac{1}{(x-1)^2} > \frac{1}{x^2} > \frac{1}{(x+1)^2}.$$

Therefore,

$$\frac{1}{(x-1)^2} + \frac{1}{x(x-1)} + \frac{1}{x^2} > \frac{1}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}$$

and

$$\frac{1}{x-1}\left(\frac{1}{(x-1)^2} + \frac{1}{x(x-1)} + \frac{1}{x^2}\right) > \frac{1}{x+1}\left(\frac{1}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}\right).$$

It follows that $\epsilon_0 = \phi(-1) > -\phi(1)$. Consequently,

$$|\phi(h)| = |f(x+h) = f(x)| < \epsilon_0 = \frac{3x^2 - 3x + 1}{x^3(x-1)^3} \qquad \text{for} \qquad |h| < 1,$$

and ϵ_0 is the "best" such tolerance in the sense that it is the smallest such tolerance.

Problem 6 (vector increments and tolerances; Boas Problem 4.4.3) The apparent distance between the image of an object and a lens is modeled by a function $I : \{(\mu, \phi) \in \mathbb{R}^2 : 0 < \phi < \mu\} \to \mathbb{R}$ as a function of the measured (model) distance μ from the object to the lens and the (model) focal length ϕ of the lens by

$$I(\mu, \phi) = \frac{\mu\phi}{\mu - \phi}.$$

Let us assume the values of $\mu_0 = 10$ and $\phi_0 = 6$ vary with a vector increment

$$(\mu, \phi) = (\mu_0, \phi_0) + (h, k)$$

where $h, k \geq 0$, but it may be ensured (or we will assume) $\mu - \phi$ is always bounded below by $m = 1$. Estimate the possible (model) change in the apparent image distance (increment)

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0)| \tag{12}$$

as follows:

(a) Draw the right triangle in $U = \{(\mu, \phi) \in \mathbb{R}^2 : 0 < \phi < \mu\}$ with vertices $(\mu_0, \phi_0) = (10, 6)$, $(10, 6 + k)$, and $(10 + h, 6 + k)$.

(b) Use the triangle inequality to show the increment in (12) is bounded above by the sum of the increments

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0 + k)| \tag{13}$$

and

$$|I(\mu_0, \phi_0 + k) - I(\mu_0, \phi_0)|. \tag{14}$$

(c) Express the increment in (14) in the form $|f(k) - f(0)|$ for an appropriate choice of $f \in C^1(0, k) \cap C^0[0, k]$, and then use the mean value theorem applied to f to get an estimate of the form

$$|I(\mu_0, \phi_0 + k) - I(\mu_0, \phi_0)| \leq G\left(\frac{\partial I}{\partial \mu}, \frac{\partial I}{\partial \phi}\right) \leq Mk$$

where the partial derivatives in the argument of the function G are evaluated at an appropriate point illustrated in your drawing from part (a) above and M has an explicit numerical value.

(d) Apply the same approach to estimating the increment in (13) to show

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0)| < Nh + Mk = \circ(1)$$

as $(h, k) \to (0, 0)$. For an explanation of the notation $\circ(1)$ used here, see the next problem.

Problem 7 (Exercise 1.26 in my notes) Recall the following definition:

Definition 1 (differentiability for a function of several variables) Given an open set $U \subset \mathbb{R}^n$ and $u : U \to \mathbb{R}$, we say u is **differentiable** at $\mathbf{p} \in U$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p}) = \circ(\|\mathbf{x} - \mathbf{p}\|)$$

as $\mathbf{x} \to \mathbf{p}$. The notation on the right here is read "little-o of $\|\mathbf{x} - \mathbf{p}\|$." It means the **limit of the quotient** of $u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})$ and $\|\mathbf{x} - \mathbf{p}\|$ is zero as $\mathbf{x}$ tends to $\mathbf{p}$, or more properly for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{p}\| < \delta \qquad \text{implies} \qquad \left| \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|} \right| < \epsilon.$$

Show differentiability implies partial differentiability. Hint: One way that $\mathbf{x}$ can limit to $\mathbf{p} \in U$ is in the form $\mathbf{x} + h\mathbf{e}_j$ as h tends to zero.

Problem 8 (partial derivatives in Problem 7 above) In Problem 7 above you should have noticed that the partial derivatives of u are given by

$$\frac{\partial u}{\partial x_j}(\mathbf{p}) = L(\mathbf{e}_j)$$

where $L = du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$ is the **differential map**. What does this tell you about the linear function $L = du_{\mathbf{p}}$? Hint: What do you know about a real valued linear map $L : \mathbb{R}^n \to \mathbb{R}$ (from linear algebra)?

Problem 9 Prove a multidimensional mean value theorem: If U is an open subset of $\mathbb{R}^n$ and $u \in C^1(U)$ and the segment

$$\Gamma = \{(1 - t)\mathbf{p} + t\mathbf{q} : 0 \le t \le 1\}$$

is a subset of U, then there is some $\mathbf{x}$ along the segment Γ for which

$$u(\mathbf{q}) - u(\mathbf{p}) = Du(\mathbf{x}) \cdot (\mathbf{q} - \mathbf{p}).$$

Problem 10 We have given a pretty careful discussion of derivatives (partial derivatives, differentiability, differential approximation, and so forth) including estimates and tolerances. In many instances, engineers use the differential approximation results above in a kind of informal manner without worrying really about how good the approximation they are using actually is. One such approximation formula is the following:

$$u(\mathbf{q}) \approx u(\mathbf{p}) + du_{\mathbf{p}}(\mathbf{q} - \mathbf{p})$$

which can also take the form(s)

$$u(\mathbf{q}) \approx u(\mathbf{p}) + Du(\mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})$$

or

$$u(\mathbf{q}) \approx u(\mathbf{p}) + \langle Du(\mathbf{p}), \mathbf{q} - \mathbf{p} \rangle.$$

Use these informal approximation formulas to answer some questions from Boas:

(a) (Boas Problem 4.4.2) Show that for n "large" and a "small,"

$$\sqrt{n+a} - \sqrt{n} \approx \frac{a}{2\sqrt{n}}.$$

Approximate $\sqrt{10^{26} + 5} - 10^{13}$.

(b) (Boas Problem 4.4.5) If resistors of $R_1 = 25$ ohms and $R_2 = 15$ ohms are connected in parallel to produce a resistance of R, approximate the resistance $\tilde{R}_2$ required of a resistor parallel to a resistor with $\tilde{R}_1 = 25.1$ ohms if the resultant resistance is still R.

(c) (Boas Problem 4.4.6) A model of a pendulum is used to approximate the (model) acceleration of gravity using the relation

$$g = u(L, T) = \frac{4\pi^2 L}{T^2}$$

where L is the model variable for the measured length of the pendulum and T is the model variable for the measured period. If the relative error in measurement of L is assumed to be 5%, i.e.,

$$\frac{L - L_{\text{actual}}}{L_{\text{actual}}} \leq 0.05,$$

and the relative error in measurement of T is assumed to be 2%, then approximately what error should one expect (in the worst case) for the model value of g (compared to an assumed actual value of the gravitational acceleration)?