Math 6702, Assignment 2

Introduction

1. (Exercise 8) What is the first order system equivalent to the ODE

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x)?$$

   Fully justify your answer.

2. (Exercise 24) Find a system of first order equations equivalent to the hyperbolic PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

§4.2 Power Series

3. (4.2.2) Find the power series expansions for
   (a) $\cos(x + y)$ and
   (b) $\sqrt{1 + xy}$.

4. The **Taylor expansion** of a function $f \in C^\infty(\mathbb{R})$ at $x_0 \in \mathbb{R}$ is given by

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j. \tag{1}$$

   Here $f^{(j)}$ denotes the j-th (ordinary) derivative of f as usual:

$$f^{(j)} = \frac{d^j f}{dx^j}.$$

   A function $f \in C^\infty(\mathbb{R})$ is said to be **real analytic** in the interval $I = (x_0 - r, x_0 + r)$ if the series in (1) converges for each $x \in I$ and

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j.$$

   The set of real analytic functions is denoted by C^ω. Verify that $\cos x$ is real analytic on $\mathbb{R}$, i.e., $\cos \in C^\omega(\mathbb{R})$.

5. Find a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^\infty(\mathbb{R}) \backslash C^\omega(\mathbb{R})$. Hint: Take $x_0 = 0$ and $f(x) \equiv 0$ for all $x \leq 0$. Then (try to) define $f(x)$ for $x > 0$ so that all the derivatives $f^{(j)}(0)$ are zero, but the values of $f(x)$ for $x > 0$ are nonzero. This is a pretty hard problem if you've never seen such a function before.

6. The **Taylor expansion** of a function $u \in C^\infty(U)$ at $\mathbf{x}_0 \in U \subset \mathbb{R}^n$ is given by

$$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta. \tag{2}$$

There are a lot of things in this expansion formula which are probably new to you. Don't freak out. First, just compare (2) to (1) and observe that these two formulas are the "same" or at least sort of the same, so (on the face of it) this is a pretty cool formula, if it has some sensible meaning—and it does. The exercise will lead you through what it means.

(a) In this expansion formula $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a **multi-index**, which simply means

$$\beta \in \mathbb{N}^n = \{(m_1, \ldots, m_n) : m_1, \ldots, m_n \in \mathbb{N}\} \qquad \text{where} \qquad \mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The derivative $D^\beta u$ denotes the partial derivative taken β_j times with respect to x_j for each $j = 1, 2, \ldots, n$:

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}.$$

The "length" of a multi-index β is defined by

$$|\beta| = \sum_{j=1}^{n} \beta_j.$$

Find all the multi-indices $\beta \in \mathbb{N}^3$ with $|\beta| = 2$.

(b) Write down all the second partials of a function $u : \mathbb{R}^3 \to \mathbb{R}$ in terms of multi-indices. Your answers should look like this:

$$D^{(2,0,0)} u = \frac{\partial^2 u}{\partial x^2}$$

and you should get five more for a total of six.

(c) Now let's back up a dimension to $\mathbb{R}^2$. The expansion for $f(x, y)$ given by Boas on page 192 has second order terms

$$\frac{1}{2!} \left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2 f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right].$$

The corresponding second order terms in (2) are

$$\sum_{|\beta|=2} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

where $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{x} = (x, y)$. To see that these are the same, you need to know the definition of the **factorial** of a multi-index, and you need to know how to take **multi-index powers** of a vector variable. Here are the definitions for $\beta \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$\beta! = \beta_1!\beta_2! \cdots \beta_n!.$$
$$\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}.$$

Show that the second order terms given by Boas for a function of two variables are the same ones you get from the formula given in (2) when $n = 2$.

7. Given an open set $U \subset \mathbb{R}^n$, a function $u \in C^\infty(U)$ is said to be **real analytic** if for each $\mathbf{x}_0 \in U$, there exists some $r > 0$ such that the series in (2) converges for each $\mathbf{x} \in B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < r\}$ and

$$u(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

for $\mathbf{x} \in B_r(\mathbf{x}_0) \cap U$. The set of **real analytic functions** on an open set $U \subset \mathbb{R}^n$ is denoted by $C^\omega(U)$. Find a function $u \in C^\infty(\mathbb{R}^n) \backslash C^\omega(\mathbb{R}^n)$.

Remark on notation: It is usual to denote the center of expansion of a power series in one variable by x_0 as in (1). For comparison of (2) to (1), we have used $\mathbf{x}_0$ as the (vector) center of expansion in the multivariable expansion. This causes a certain inconvenience when writing down the coordinates in higher dimensions. For $n = 2$ as in part (c) of problem 6, one can use $\mathbf{x}_0 = (x_0, y_0)$, and this approach can work for $n = 3$ as well with $\mathbf{x}_0 = (x_0, y_0, z_0)$. For general n, however, one usually resorts to something unpleasant like

$$\mathbf{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0).$$

To further understand the unpleasantness of this expression for the coordinates, you may write out the multi-index power $\mathbf{x}_0^\beta$. My preferred alternative is to replace $\mathbf{x}_0$ with $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, though some continuity of notation is lost between (1) and (2).

8. Repeat Boas' Problem 4.2.2 (given above as Problem 3) using the multi-index Taylor expansion formula.