Assignment 2: Partial derivatives
Due Friday, January 31, 2025

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Problem 1 Draw the set

$$\{\mathbf{x} + h\mathbf{e}_j : |h| < \delta\} \subset \mathbb{R}^n$$

when

(a) $n = 2$, $\mathbf{x} = (2, 1)$ and $\delta = 1/3$.

(b) $n = 3$, $\mathbf{x} = (1, 0, 1/2)$ and $\delta = 1/4$.

Problem 2 (convexity) Recall the following definition: A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1 - t)x_1 + tx_2) \le (1 - t)f(x_1) + tf(x_2) \tag{1}$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \le t \le 1$.

Show that if f is convex, then

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{2}$$

Hint: Assume there is some $\epsilon > 0$ and a sequence of points $x_1, x_2, x_3, \ldots$ with $x_j \nearrow x_0$ and $f(x_j) \le f(x_0) - \epsilon$. If you can get a contradiction out of this, it means

$$\lim_{x \nearrow x_0} f(x) \ge f(x_0).$$

What does (2) tell you about convex functions?

Problem 3 Review the definitions in Problem 8 (continuity) and Problem 9 (differentiability) of Assignment 1. Show that if a function $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x \in (a, b)$, then f is continuous at x.

Problem 4 Draw a picture of the **graph**

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in A\}$$

where $A = (0, 1) \times (0, 2)$ is an open rectangle and

$$u(x, y) = 1 + x^2.$$

Draw in red the curves

$$\{(1/2 + t, 1, u(1/2 + t, 1)) : 0 \le t \le 1/2\} \subset \mathcal{G}$$

and

$$\{(1/2 + t, u(1/2 + t, 1)) \in \mathbb{R}^2 : 0 \le t \le 1/2\}.$$

Illustrate the (two) difference quotients

$$\frac{u(1/2 + h, 1) - u(1/2, 1)}{h} \qquad \text{and} \qquad \frac{u(1/2, 1 + h) - u(1/2, 1)}{h}$$

for u at $(1/2, 1)$ with $h = 1/4$.

Problem 5 Give an example to show that the existence of the partial derivatives

$$\frac{\partial u}{\partial x_j}(\mathbf{x}) = \lim_{h \to 0} \frac{u(\mathbf{x} + h\mathbf{e}_j) - u(\mathbf{x})}{h}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and each $j = 1, \ldots, n$, i.e., partial differentiability, does not imply continuity (when $n > 1$).

Problem 6 (increments and tolerances; Boas Problem 4.4.1) Consider $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^3}.$$

Note that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = -\frac{3}{x^4}.$$

(a) Given $x > 0$ and a tolerance $\epsilon_1 > 0$, find a tolerance $\delta_1 > 0$ so that

$$0 < |h| < \delta_1 \qquad \text{implies} \qquad \left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| < \epsilon_1. \tag{3}$$

Hint: Assume $\delta_1 < \min\{x/2, \delta\}$ where δ is some other number. Use the estimate $\delta_1 < x/2$ to simplify the "extraneous" algebraic expression you obtain from simplifying the quantity to be estimated in (3). Then determine how small you need to make δ. Your answer should depend on ϵ_1 and x.

(b) Given $x > 0$ and a tolerance $\epsilon_0 > 0$, find a tolerance δ_0 for which

$$0 < |h| < \delta_0 \qquad \text{implies} \qquad \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| < \epsilon_0. \tag{4}$$

Hint: Use the same approach as in part **(a)**; but this one is easier.

(c) Say you can't pick the tolerance δ_0 in part **(b)**, but you are stuck with $|h| \le \delta_0 = 1$. What is the best tolerance ϵ_0 you can get in (4)?

Problem 7 A function $f : \mathbb{C} \to \mathbb{C}$ with real and imaginary parts expressed as functions $u, v \in C^2(\mathbb{R}^2)$ so that $f(x+iy) = u(x,y) + v(x,y)\, i$ is said to be **complex differentiable** at $z = x + iy \in \mathbb{C}$ if

$$f'(z) = \lim_{h \to 0+0i} \frac{f(z+h) - f(z)}{h}$$

exists. Show that if f is complex differentiable at every $z = x + iy \in \mathbb{C}$, then the functions u and v satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Hint: Take the limit first in the special case where $h = \epsilon \in \mathbb{R}$. Then take the limit in the special case where $h = \epsilon\, i$. In each case you should get answers involving first order partial derivatives of u and v. Because the answers you get look different, but according to the definition of what it means to be complex differentiable must be the same, you should be able to obtain some interesting relations among the first partial derivatives.

Problem 8 (Exercise 1.26 in my notes) Recall the following definition:

Definition 1 (differentiability for a function of several variables) Given an open set $U \subset \mathbb{R}^n$ and $u : U \to \mathbb{R}$, we say u is **differentiable** at $\mathbf{p} \in U$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p}) = \circ(\|\mathbf{x} - \mathbf{p}\|)$$

as $\mathbf{x} \to \mathbf{p}$. The notation on the right here is read "little-o of $\|\mathbf{x} - \mathbf{p}\|$." It means the **limit of the quotient** of $u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})$ and $\|\mathbf{x} - \mathbf{p}\|$ is zero as $\mathbf{x}$ tends to $\mathbf{p}$, or more properly for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{p}\| < \delta \qquad \text{implies} \qquad \left| \frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|} \right| < \epsilon.$$

Show differentiability implies partial differentiability. Hint: One way that $\mathbf{x}$ can limit to $\mathbf{p} \in U$ is in the form $\mathbf{x} + h\mathbf{e}_j$ as h tends to zero.

Problem 9 (partial derivatives in Problem 7 above) In Problem 7 above you should have noticed that the partial derivatives of u are given by

$$\frac{\partial u}{\partial x_j}(\mathbf{p}) = L(\mathbf{e}_j)$$

where $L = du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$ is the **differential map**. What does this tell you about the linear function $L = du_{\mathbf{p}}$? Hint: What do you know about a real valued linear map $L : \mathbb{R}^n \to \mathbb{R}$ (from linear algebra)?

Problem 10 Note the following:

(i) In Problem 5 above you showed partial differentiability does not imply continuity.

(ii) In Problem 8 above you showed differentiability implies partial differentiability.

Can you show differentiability implies continuity?