Assignment 2: Partial derivatives
Due Wednesday, February 1, 2023

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Problem 1 Draw the set

$$\{\mathbf{x} + h\mathbf{e}_j : |h| < \delta\} \subset \mathbb{R}^n$$

when

(a) $n = 2$, $\mathbf{x} = (2, 1)$ and $\delta = 1/3$.

(b) $n = 3$, $\mathbf{x} = (1, 0, 1/2)$ and $\delta = 1/4$.

Problem 2 (convexity) Recall the following definition: A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2) \qquad (1)$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \le t \le 1$.

Show that if f is convex, then

$$\lim_{x \to x_0} f(x) = f(x_0). \qquad (2)$$

Hint: Assume there is some $\epsilon > 0$ and a sequence of points $x_1, x_2, x_3, \ldots$ with $x_j \nearrow x_0$ and $f(x_j) \le f(x_0) - \epsilon$. If you can get a contradiction out of this, it means

$$\lim_{x \nearrow x_0} f(x) \ge f(x_0).$$

What does (2) tell you about convex functions?

Problem 3 Draw a picture of the **graph**

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in A\}$$

where $A = (0, 1) \times (0, 2)$ is an open rectangle and

$$u(x, y) = 1 + x^2.$$

Draw in red the curves

$$\{(1/2 + t, 1, u(1/2 + t, 1)) : 0 \leq t \leq 1/2\} \subset \mathcal{G}$$

and

$$\{(1/2 + t, u(1/2 + t, 1)) \in \mathbb{R}^2 : 0 \leq t \leq 1/2\}.$$

Illustrate the (two) difference quotients

$$\frac{u(1/2 + h, 1) - u(1/2, 1)}{h} \qquad \text{and} \qquad \frac{u(1/2, 1 + h) - u(1/2, 1)}{h}$$

for u at $(1/2, 1)$ with $h = 1/4$.

Problem 4 Here is a definition of what it means for a subset of $\mathbb{R}^n$ to be open: A set $U \subset \mathbb{R}^n$ is **open** if for each

$$\mathbf{p} = \sum_{j=1}^{n} p_j \mathbf{e}_j \in U,$$

there is some $\delta > 0$ for which

$$Q_\delta(\mathbf{p}) = \left\{ \mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j : |x_j - p_j| < \frac{\delta}{2} \right\} \subset U.$$

The set $Q_\delta(\mathbf{p})$ is called the **open cube of side length δ centered at $\mathbf{p}$**.

Show that for $\delta > 0$ and $\mathbf{p} \in \mathbb{R}^n$, the cube $Q_\delta(\mathbf{p})$ is open.

Problem 5 (increments and tolerances; Boas Problem 4.4.1) Consider $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{1}{x^3}.$$

Note that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = -\frac{3}{x^4}.$$

(a) Given $x > 0$ and a tolerance $\epsilon_1 > 0$, find a tolerance $\delta_1 > 0$ so that

$$0 < |h| < \delta_1 \qquad \text{implies} \qquad \left| \frac{f(x+h) - f(x)}{h} + \frac{3}{x^4} \right| < \epsilon_1. \tag{3}$$

Hint: Assume $\delta_1 < \min\{x/2, \delta\}$ where δ is some other number. Use the estimate $\delta_1 < x/2$ to simplify the "extraneous" algebraic expression you obtain from simplifying the quantity to be estimated in (3). Then determine how small you need to make δ. Your answer should depend on ϵ_1 and x.

(b) Given $x > 0$ and a tolerance $\epsilon_0 > 0$, find a tolerance δ_0 for which

$$0 < |h| < \delta_0 \qquad \text{implies} \qquad \left| \frac{1}{(x+h)^3} - \frac{1}{x^3} \right| < \epsilon_0. \tag{4}$$

Hint: Use the same approach as in part **(a)**; but this one is easier.

(c) Say you can't pick the tolerance δ_0 in part **(b)**, but you are stuck with $|h| \le \delta_0 = 1$. What is the best tolerance ϵ_0 you can get in (4)?

Problem 6 (vector increments and tolerances; Boas Problem 4.4.3) The apparent distance between the image of an object and a lens is modeled by a function $I : \{(\mu, \phi) \in \mathbb{R}^2 : 0 < \phi < \mu\} \to \mathbb{R}$ as a function of the measured (model) distance μ from the object to the lens and the (model) focal length ϕ of the lens by

$$I(\mu, \phi) = \frac{\mu\phi}{\mu - \phi}.$$

Let us assume the values of $\mu_0 = 10$ and $\phi_0 = 6$ vary with a vector increment

$$(\mu, \phi) = (\mu_0, \phi_0) + (h, k)$$

where $h, k \ge 0$, but it may be ensured (or we will assume) $\mu - \phi$ is always bounded below by $m = 1$. Estimate the possible (model) change in the apparent image distance (increment)

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0)| \tag{5}$$

as follows:

(a) Draw the right triangle in $U = \{(\mu, \phi) \in \mathbb{R}^2 : 0 < \phi < \mu\}$ with vertices $(\mu_0, \phi_0) = (10, 6)$, $(10, 6 + k)$, and $(10 + h, 6 + k)$.

(b) Use the triangle inequality to show the increment in (5) is bounded above by the sum of the increments

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0 + k)| \tag{6}$$

and

$$|I(\mu_0, \phi_0 + k) - I(\mu_0, \phi_0)|. \tag{7}$$

(c) Express the increment in (7) in the form $|f(k) - f(0)|$ for an appropriate choice of $f \in C^1(0, k) \cap C^0[0, k]$, and then use the mean value theorem applied to f to get an estimate of the form

$$|I(\mu_0, \phi_0 + k) - I(\mu_0, \phi_0)| \le G\left(\frac{\partial I}{\partial \mu}, \frac{\partial I}{\partial \phi}\right) \le Mk$$

where the partial derivatives in the argument of the function G are evaluated at an appropriate point illustrated in your drawing from part (a) above and M has an explicit numerical value.

(d) Apply the same approach to estimating the increment in (6) to show

$$|I(\mu_0 + h, \phi_0 + k) - I(\mu_0, \phi_0)| < Nh + Mk = \circ(1)$$

as $(h, k) \to (0, 0)$. For an explanation of the notation $\circ(1)$ used here, see the next problem.

Problem 7 (Exercise 1.26 in my notes) Recall the following definition:

Definition 1 (differentiability for a function of several variables) Given an open set $U \subset \mathbb{R}^n$ and $u : U \to \mathbb{R}$, we say u is **differentiable** at $\mathbf{p} \in U$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p}) = \circ(\|\mathbf{x} - \mathbf{p}\|)$$

as $\mathbf{x} \to \mathbf{p}$. The notation on the right here is read "little-o of $\|\mathbf{x} - \mathbf{p}\|$." It means the **limit of the quotient** of $u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})$ and $\|\mathbf{x} - \mathbf{p}\|$ is zero as $\mathbf{x}$ tends to $\mathbf{p}$, or more properly for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{p}\| < \delta \qquad \text{implies} \qquad \left|\frac{u(\mathbf{x}) - u(\mathbf{p}) - L(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|}\right| < \epsilon.$$

Show differentiability implies partial differentiability. Hint: One way that $\mathbf{x}$ can limit to $\mathbf{p} \in U$ is in the form $\mathbf{x} + h\mathbf{e}_j$ as h tends to zero.

Problem 8 (partial derivatives in Problem 7 above) In Problem 7 above you should have noticed that the partial derivatives of u are given by

$$\frac{\partial u}{\partial x_j}(\mathbf{p}) = L(\mathbf{e}_j)$$

where $L = du_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}$ is the **differential map**. What does this tell you about the linear function $L = du_{\mathbf{p}}$? Hint: What do you know about a real valued linear map $L : \mathbb{R}^n \to \mathbb{R}$ (from linear algebra)?

Problem 9 Prove a multidimensional mean value theorem: If U is an open subset of $\mathbb{R}^n$ and $u \in C^1(U)$ and the segment

$$\Gamma = \{(1-t)\mathbf{p} + t\mathbf{q} : 0 \le t \le 1\}$$

is a subset of U, then there is some $\mathbf{x}$ along the segment Γ for which

$$u(\mathbf{q}) - u(\mathbf{p}) = Du(\mathbf{x}) \cdot (\mathbf{q} - \mathbf{p}).$$

Problem 10 We have given a pretty careful discussion of derivatives (partial derivatives, differentiability, differential approximation, and so forth) including estimates and tolerances. In many instances, engineers use the differential approximation results above in a kind of informal manner without worrying really about how good the approximation they are using actually is. One such approximation formula is the following:

$$u(\mathbf{q}) \approx u(\mathbf{p}) + du_{\mathbf{p}}(\mathbf{q} - \mathbf{p})$$

which can also take the form(s)

$$u(\mathbf{q}) \approx u(\mathbf{p}) + Du(\mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})$$

or

$$u(\mathbf{q}) \approx u(\mathbf{p}) + \langle Du(\mathbf{p}), \mathbf{q} - \mathbf{p} \rangle.$$

Use these informal approximation formulas to answer some questions from Boas:

(a) (Boas Problem 4.4.2) Show that for n "large" and a "small,"

$$\sqrt{n+a} - \sqrt{n} \approx \frac{a}{2\sqrt{n}}.$$

Approximate $\sqrt{10^{26} + 5} - 10^{13}$.

(b) (Boas Problem 4.4.5) If resistors of $R_1 = 25$ ohms and $R_2 = 15$ ohms are connected in parallel to produce a resistance of R, approximate the resistance $\tilde{R}_2$ required of a resistor parallel to a resistor with $\tilde{R}_1 = 25.1$ ohms if the resultant resistance is still R.

(c) (Boas Problem 4.4.6) A model of a pendulum is used to approximate the (model) acceleration of gravity using the relation

$$g = u(L, T) = \frac{4\pi^2 L}{T^2}$$

where L is the model variable for the measured length of the pendulum and T is the model variable for the measured period. If the relative error in measurement of L is assumed to be 5%, i.e.,

$$\frac{L - L_{\text{actual}}}{L_{\text{actual}}} \leq 0.05,$$

and the relative error in measurement of T is assumed to be 2%, then approximately what error should one expect (in the worst case) for the model value of g (compared to an assumed actual value of the gravitational acceleration)?