Assignment 1: Solutions
Due Friday, January 24, 2025

John McCuan

January 24, 2025

Problem 1 (modeling a hanging slinky—step zero) Determine/identify some (interesting) quantity associated with a hanging slinky which you think can be measured and modeled by, i.e., compared to, a real valued function

$$f : (a, b) \to \mathbb{R}$$

of one variable on an open interval (a, b) or possibly on a closed interval $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$. You should identify the real numbers a and b with $a < b$ determining the **interval of definition** though measurements may be needed to give them actual numerical values, and identify the quantity or measurement to which **the values** of the function f should be compared.

When you get done, you should have an idea of exactly what you want to measure and how.

You may wish to change the name of the function f. For example, if you want to compare the values of f to a linear density, then you may want to call the function ρ, λ, or δ, which are more traditional symbols used to denote a linear density. Hint: Do not let f correspond to/model a linear density but rather some quantity which is easier to measure and from which a linear density may be derived.

Let's call your function a **model measurement function**.

Solution: It seems to me the domain for modeling the hanging slinky, if it is going to be an interval, should be some interval associated with the geometry of the slinky when it is totally compressed as it might be if you rest it on a table or as it sits in the box when new. There are, it seems to me a couple obvious choices. One is simply an interval $(0, h_0)$ or $[0, h_0]$ where h_0 is the "height" of the compressed slinky.

By **compressed**, I mean in the position where all the coils are stacked directly together with "negligible" distance between coils. This is probably a reasonable model assumption and it is probably reasonable to attach to it the assumption that this configuration represents some kind of equilibrium for the slinky. This is of course one property that distinguishes a slinky from the usual spring modeled as some kind of one dimensional elastic continuum: Usually when an elastic continuum is extended beyond the equilibrium position a restoring tension force is assumed to result, and this seems appropriate for the slinky also, but it is also usually assumed that when an elastic continuum is compressed, then some kind of proportional restoring expansion force results. This latter doesn't really happen (too much) with a slinky because if the coils are stacked tightly together in equilibrium, then further compression is not geometrically or mechanically possible.

In reality I don't think I have ever seen a classic metal slinky that stacks together perfectly. The slinky I have sitting on my desk now will compress a little bit if I push down on it. The compression is not much, and perhaps it is reasonable to consider it negligible in the model, but it is clearly there. If I measure the height of the slinky sitting on my desk with a caliper, I get 52.6 mm. If I manually compress the slinky with the caliper, I see a reading between 50.7 mm and 51.3 mm which I would be inclined to take as the appropriate measurement for the value h_0. My caliper only reads to the nearest 0.1 mm, so I think I can be fairly confident with the value 51 mm.

It occurs to me there may be an interesting way to test this model assumption about the coils being stacked. It is possible to measure the (vertical) thickness of a single coil. There turns out to be some variation. Most of the coils clearly read to have thickness 0.6 mm for the slinky in front of me, but a few of them are clearly thicker than 0.6 mm. I don't find any larger than 0.7 mm. I can also count the number of coils, and I get 78 full coils with about three-quarters of an extra coil. Thus the height h_0 I am measuring is for 79 thicknesses of coil. With this information I can calculate some initial bounds for the measurement h_0:

$$47.4 \text{ mm} < h_0 < 55.3 \text{ mm}.$$

Taking $h_0 \doteq 51$ and assuming a discrete distribution of n coils with thickness 0.6 mm and $79 - n$ coils with thickness 0.7 mm implies there should be 43 thin coils and 36 thick ones. I wanted to check. What I found was that there are far fewer than 36 thick coils. There are only about 10 and certainly not more than 15. Thus, the mere presence of some thick coils did not seem to account for the overall height of 51 mm. It seemed there was somewhere between 2 mm and 4 mm that needed some explanation. I think an explanation can be found in two or three observations or propositions. The first suggestion is to disperse the unexplained height over the large number of coils. Thus, if there are 4 mm of "extra" height, then this means that (perhaps) we are just picking up about 0.05 mm in the stacking of each individual coil. Second, the reading of the caliper is only at best to the nearest 0.1 mm, meaning that when we read 0.6 mm it is reasonable to imagine this might actually be representative of something very close to 0.64 mm. On the one hand, picking up 0.04 mm in each reading and adding in the handful of extra thick coils, could account for 4 mm of extra height, but this is probably not quite the full and correct explanation. The final factor I would suggest is that the coils do not actually stack perfectly, but there are likely very small "gaps" between the coils caused by a slight level of waviness in the coils. Rather than making contact at all points, probably there are certain points where the metal of one coil contacts the coils next to it with very small air "gaps" in between those points. Overall, this non-perfect stacking probably accounts well for the surprisingly large height h_0 compared to the sum of the individual thicknesses of the coils.

Okay, that has maybe taken us a little bit astray, but I've offered one possibility for the domain of the model measurement function: The interval $[0, h_0]$ where $h_0 \doteq 51$ mm is the height of the slinky when "completely compressed."

A natural second and perhaps more fundamental and precise modeling possibility is an interval modeling the length ℓ_0 of the entire coil. To use this domain properly requires a little more work. I will start by making more measurements. The width of the rectangular cross-section of the coil measures 2.5 mm. The outer diameter of the coils measures 66.6 mm and the inner diameter measures 61.5 mm. Those add up nicely well within a 0.1 mm tolerance. Taking the average diameter of 64 mm and the more precise number of 78.64 coils, consider a "core curve" parameterized by $\alpha : (0, 2\pi(78.64)) \to \mathbb{R}^3$ with

$$\alpha(\theta) = 0.032(\cos\theta, \sin\theta, 0) + (a\theta + d)\mathbf{e}_3.$$

where let's say we take the half-thickness of the coils to be $d \doteq 0.3$ mm $= 0.0003$ m and assume that value is uniform. We know the thickness is not quite uniform, but for the moment let us take this as an approximation. Similarly, we can assume the core curve we have corresponds to a kind of perfect stacking on a table ($z = 0$), and we can be pretty sure this is not exactly what is observed or measured, but we will take it as a reasonable approximation. Under these assumptions we should get

$$2\pi a(78) + 2d = h_0$$

for the measured height h_0, so we have an initial "perfect stacking" value for a, namely

$$a = \frac{h_0 - 2d}{2\pi(78)}.$$

Using this value, we can reparameterize the core curve by arclength and find the total length ℓ_0 (if we know how). The (arc)length along the core curve is given by

$$s = \int_0^\theta |\alpha'(t)|\, dt = \sqrt{r_0^2 + a^2}\ \theta$$

where $r_0 \doteq 0.032$ is the core radius.

Thus, a paremeterization by arclength $\gamma : [0, \ell_0] \to \mathbb{R}^3$ of the core curve satisfying $|\dot{\gamma}| = 1$ is given by

$$\gamma(s) = r_0 \left(\cos \frac{s}{\sqrt{r_0^2 + a^2}}, \sin \frac{s}{\sqrt{r_0^2 + a^2}}, 0 \right) + \left(\frac{as}{\sqrt{r_0^2 + a^2}} + d \right) \mathbf{e}_3,$$

and

$$\ell_0 = \theta_0 \sqrt{r_0^2 + a^2} \qquad \text{where} \qquad \theta_0 \doteq 2\pi(78.64).$$

Substituting the measured values we obtain a length $\ell_0 \doteq 15.812$ m. The equilibrium height h_0 of the entire slinky and the equilibrium position/height of each coil may then be considered derived quantities as functions of s with $0 < s < \ell_0$. This second domain also provides us with a foundation for a model that gives the position in three-dimensional space for various points on the slinky. In particular, γ gives a kind of idealized equilibrium position, and gives a starting point for further examination of two interesting questions raised above about the equilibrium:

1. Can one model the evenly distributed "non-perfect stacking" of the slinky resulting in the 2 mm to 4 mm difference between the sum of the thicknesses of the coils $79d$ and the actual measured height $h_0 \doteq 32$ mm?

2. Can one model the 2 mm difference in the height of the slinky measured sitting in physical equilibrium on a table in reference to the measurement of the height $h_0 \doteq 32$ mm measured under active compression?

Another thing one can do is create some graphics illustrating the model. In Figure 1 I have drawn the model image associated with the first coil of the slinky sitting on the table. If you don't understand the formula for α or γ, then hopefully you can use this picture to help. For example, maybe you can try to create such an illustration on your own.

Say for example, you didn't understand why I counted 78 coils but used 79 times the thickness to get the measurement for h_0. I can now make graphics to explain

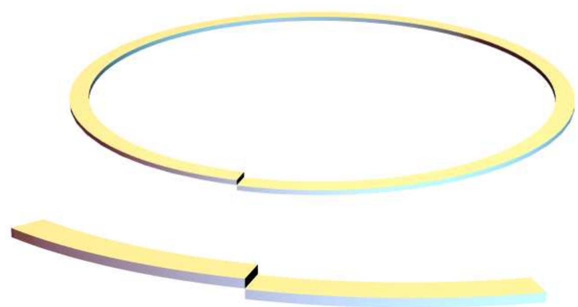

Figure 1: The first coil.

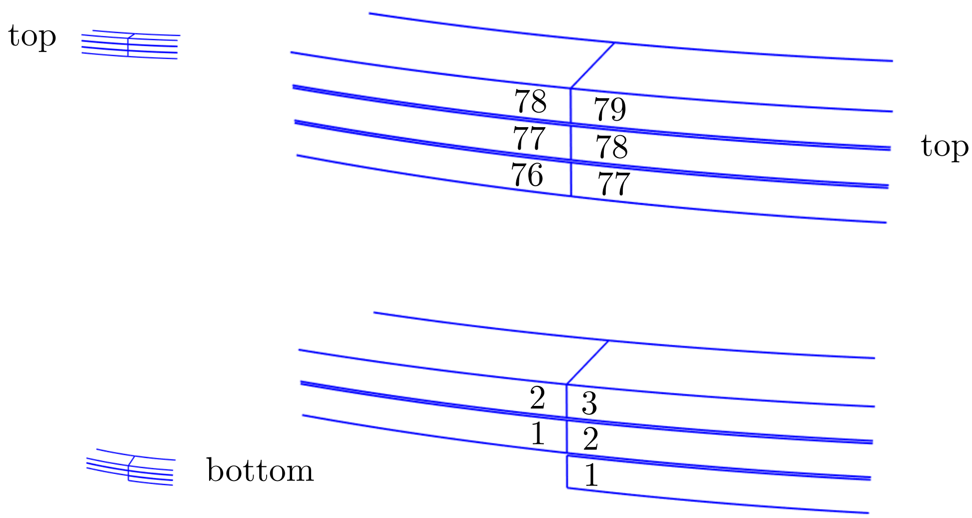

Figure 2: Measuring height.

that point. In Figure 2 I've drawn the first few coils and the last few coils as they appear near the point where the measurement of h_0 is made. Notice that if there were only two full coils and part of a third, then the height from the table would be three thicknesses of the coils. This pattern continues, so that 78 full coils and part of one more coil gives a caliper measurement of 79 times the thickness d.

As a final benefit of the length model for the domain, we can see the height model for the domain can (and perhaps should) be improved. It is certainly possible to use the interval $[0, h_0]$ or $(0, h_0)$ as the model domain. It is natural, however, to have each point in a model domain correspond to a "material point" in the real world object one is attempting to model. We accomplish this pretty well with the core curve in the length model. Notice that not every point in the physical spring is represented by a point on the core curve necessarily, but certainly each point in the core curve corresponds to some point within the material of the slinky, and in a kind of uniform manner, except perhaps for the endpoints 0 and ℓ of $[0, \ell]$. If we wish to achieve the same correspondence with a height interval, we should replace $[0, h_0]$ with the image of the third component of γ, namely

$$[\gamma_3(0), \gamma_3(\ell_0)] = [d, h_1 - d] \doteq [0.0003, 0.0511]$$

where $h_1 \doteq 0.0514$ m $= 51.4$ mm is the height above the table of the very last portion of coil after the seventy-eighth coil.

Having discussed two (or three) possible model domains, it seems a natural model codomain for the hanging slinky is $[0, \infty)$ where distance from the point of suspension is measured. Specifically, if we introduce coordinates on $\mathbb{R}^3$ so the point of suspension is modeled by the origin $\mathbf{0} = (0, 0, 0)$, say at the center of the coil attached to the plate, if that is well-defined. Then assuming there is a unique material point in the slinky determined by x in the domain we assume this material point is located at $\mathbf{X} = (X_1, X_2, X_3)$ when the slinky is turned upside down and suspended. We assume the height X_3 is uniquely determined by x and take $-X_3$ as the value of a model measurement function in $[0, \infty)$.

Consider the three cases separately. In call cases, we can call the model measurement function σ the **stretch**.

I. length model We consider a function $\mathbf{X} : [0, \ell_0] \to \mathbb{R}^3$ with

$$\mathbf{X}(s) = (X_1(s), X_2(s), X_3(s)) = r(\cos\theta, \sin\theta, 0) + X_3\mathbf{e}_3$$

modeling the material location in the hanging spring of the material point in the equilibrium spring corresponding to $\gamma(s)$. Then the main model measurement function is a stretch function $\sigma : [0, \ell_0] \to [0, \infty)$ by

$$\sigma(s) = -X_3(s).$$

Notice that in this case the additional model measurement functions $r = r(s)$, $\theta = \theta(s)$, and $X_3 = X_3(s)$ have been introduced and are available for consideration.

II. height model (improved) We consider a stretch function $\sigma : [d, h_1 - d] \to [0, \infty)$ where as above $\sigma(h) = -X_3$ is the vertical distance from the material point $\mathbf{X} = (X_1, X_2, X_3)$ to the point $(0, 0, 0)$ of suspension of the hanging spring.

III. height model (original) We consider a stretch function $\sigma : [0, h_0] \to [0, \infty)$ where $\sigma(h) = -X_3$ is the vertical distance from the material point $\mathbf{X} = (X_1, X_2, X_3)$ to the point $(0, 0, 0)$ of suspension of the hanging spring.

Just as a preliminary indication that we have a reasonable model measurement function, we can take a couple, more or less arbitrary, choices for $\mathbf{X}$ and see if we get the results we expect. This is not really modeling per se, but just an exercise to make sure the model measurement function(s) we have are making sense. First of all, we can imagine a hanging slinky that, for some inexplicable reason hangs down in a uniformly extended fashion. I'm choosing this just becuase the formula is relatively easy to write down. In Figure 3 I've put three pictures of the slinky core curve sitting ideally compressed (or perfectly stacked) in equlibrium. In the right figure, I have put a marker on the front point $(r_0, 0, 2\pi ak + d)$ of the core curve for each $k = 0, 1, 2, \ldots, 78$. We can keep track of these markers for various deformations from the equilibrium position.

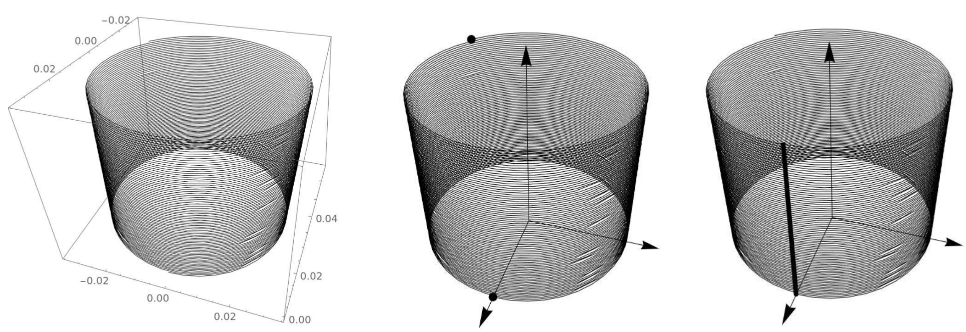

Figure 3: Equilibrium model. Core curve (left). Endpoints marked (middle). Front points marked (right).

Figure 4 illustrates the slinky uniformly extended downward with the associated stretch function plotted as a graph and illustrated as a mapping. One may get a nicer illustration of the mapping if only every third or every sixth front point is given representation in the graphic. Obviously, we expect the actual measurements we want to compare to the model measurement function σ to be very different from the values

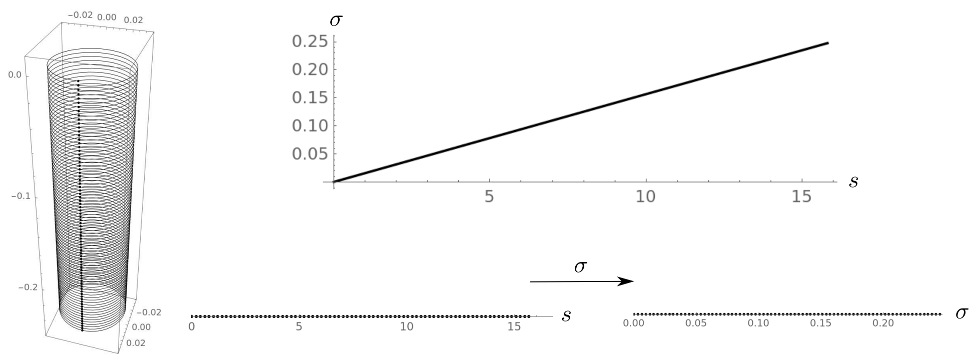

Figure 4: "Hanging" configuration with uniform extension downward. (Consider manual extension in zero gravity.) Underlying geometry (left). Stretch σ as a function of arclength s along the coil: Graph (top right). Mapping (bottom right).

illustrated in Figure 4; the total extension here is only about twelve inches, and we know the extension should be greater near the top and much smaller near the bottom. In short, this arbitrary function σ does not match the data very well at all.

We can get some nominal improvement simply by finding a function X that represents greater extension at the top and smaller extension near the bottom. Such a choice is illustrated in Figure 5 where we have also used instead the model domain $[d, h_1 - d]$ so the stretch σ effectively maps equilibrium heights h to hanging depths σ. This function σ is still quite arbitrary, the agreement with actual measurements cannot be expected to be very good, and it cannot be said that we have done any actual mathematical modeling of the system at this point. We simply have a reasonable model measurement function.

The final stretch function $\sigma : [0, h_0] \to [0, \infty)$ of category **III** is, on the one hand, perhaps the most natural one to consider as a primitive model measurement function, and on the other hand, as pointed out above, this model measurement function has certain shortcomings. The most obvious is that there is not a nice correspondence between the domain interval $[0, h_0]$ and specific material points in the slinky. It turns out that the assumed relation, or more properly approximate relation, among this stretch function and the other two is somewhat subtle and problematic. In any case, taking this choice of σ is compelling and convenient in many ways, and I suggest this as a good start and also postponing further discussion of the possible relations among the stretch functions of **I**, **II**, and **III**. The consideration of relations between the stretch function $\sigma : [0, h_0] \to [0, \infty)$ with $\sigma(0) = 0$ of category **III** with the other

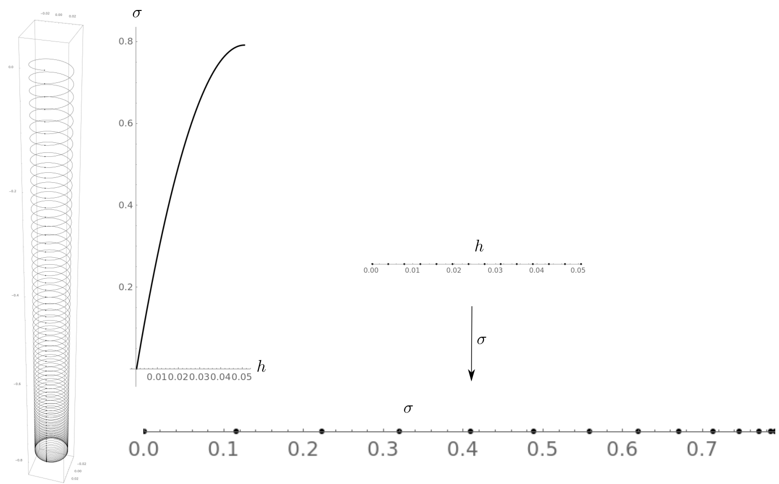

Figure 5: "Hanging" configuration with nonuniform extension downward. Underlying geometry (left). Stretch σ as a function of equilibrium height h: Graph (middle). Mapping (right/bottom).

two does seem to lead to some relatively interesting aspects of the problem that are perhaps worthwhile to take up later. If you wish to think about this relation, perhaps something to keep in mind (or at least to think about) is how accurately you might expect to measure the values of stretch on the actual physical slinky.

Problem 2 (convexity) Find a convex function $f : (-2, 1) \to \mathbb{R}$ satisfying

$$\lim_{x \searrow -2} f(x) = \lim_{x \nearrow 1} f(x) = +\infty.$$

A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2) \tag{1}$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \le t \le 1$.

Solution: Such a function is given by

$$f(x) = \sec \frac{\pi}{3}\left(x + \frac{1}{2}\right) = \frac{1}{\cos \frac{\pi}{3}\left(x + \frac{1}{2}\right)}.$$

Note that $\xi : \mathbb{R} \to \mathbb{R}$ by

$$\xi(x) = \frac{\pi}{3}\left(x + \frac{1}{2}\right)$$

is increasing with $\xi(-2) = -\pi/2$ and $\xi(1) = \pi/2$.

This is a function like many others which can be shown to be convex because it is twice continuously differentiable with positive second derivative. Let $f : (a, b) \to \mathbb{R}$ be any such function and for $a < x_1 < x_2 < b$ consider $\phi : (0, 1) \to \mathbb{R}$ by

$$\phi(t) = (1-t)f(x_1) + tf(x_2) - f((1-t)x_1 + tx_2).$$

Notice that ϕ actually extends to be continuous on $[0, 1]$ with $\phi(0) = \phi(1) = 0$. Also, computing directly we see

$$\phi'(t) = (x_2 - x_1)\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'((1-t)x_1 + tx_2)\right),$$

and

$$\phi''(t) = -(x_2 - x_1)f''((1-t)x_1 + tx_2) < 0. \tag{2}$$

If we assume by way of contradiction that $\phi(t) \le 0$ for some t with $0 < t < 1$, then there exists an interior minimum $\phi(t_0) \le \phi(t)$ for all $0 \le t \le 1$ with $0 < t_0 < 1$ and $\phi(t_0) \le 0$. The necessary conditions for an interior minimum then give

$$\phi'(t_0) = 0 \qquad \text{and} \qquad \phi''(t_0) \ge 0.$$

The latter inequality contradicts (2). Thus, $\phi(t) > 0$ for $0 < t < 1$, and this means f is convex.

In our case

$$\begin{aligned} f'(x) &= \frac{\pi}{3} \sec\xi \tan\xi, \\ f''(x) &= \frac{\pi^2}{9} \left[\sec\xi \tan^2\xi + \sec^3\xi\right] \\ &= \frac{\pi^2}{9} \sec\xi \left[\tan^2\xi + \sec^2\xi\right] \\ &> 0 \end{aligned}$$

for $-2 < x < 1$ since $\sec\xi = 1/\cos\xi$ and $\cos\xi > 0$ for $|\xi| < \pi/2$.

Finally to see the limits, we note simply that

$$\lim_{x \searrow -2} f(x) = \lim_{\xi \searrow -\pi/2} \frac{1}{\cos\xi} = +\infty$$

and

$$\lim_{x \nearrow -2} f(x) = \lim_{\xi \nearrow \pi/2} \frac{1}{\cos\xi} = +\infty.$$

Problem 3 Draw a picture of the **graph**

$$\{(x, f(x)) : x \in (a, b)\}$$

of a convex function $f : (a, b) \to \mathbb{R}$ illustrating the condition (1).

Solution:

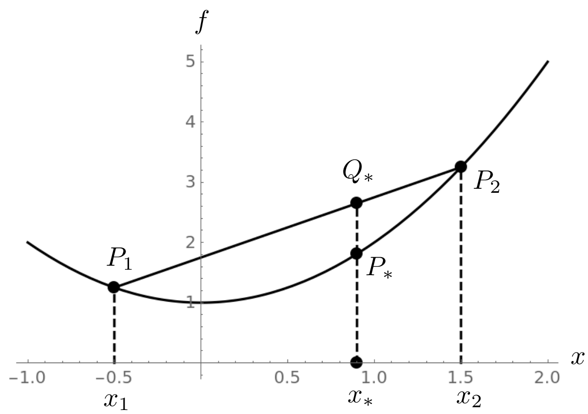

Figure 6: A convex function, a secant line, and convex combinations.

In the example of Figure 6 the function $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2 + 1$. There are labeled points $P_j = (x_j, f(x_j))$ for $j = 1, 2$ and a point $P_* = (x_*, f(x_*))$ with $x_* = (1-t)x_1 + tx_2$ for some t with $0 < t < 1$. The point on the secant line labeled Q_* is

$$Q_* = (1-t)P_1 + tP_2 = (x_*, (1-t)f(x_1) + tf(x_2)).$$

Problem 4 Is it possible to find an example of a convex function $f : (a, b) \to \mathbb{R}$ that is discontinuous?

Solution: The answer to this question is "no." Every convex function on an open interval is continuous. There are various ways to show this. Perhaps one of the easiest ways in this case is the following:

Proceed by contradiction and assume there is a convex function $f : (a, b) \to \mathbb{R}$ which is discontinuous at a point $x_0 \in (a, b)$. This means there exists some $\epsilon_0 > 0$ such that no matter which $\delta > 0$ is chosen, there still exists a point x with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon_0$. Using this assumption, we can take $\delta_j = 1/j$ for $j = 1, 2, \ldots$ and obtain a sequence of points x_j with

$$|x_j - x_0| < \frac{1}{j} \qquad \text{and} \qquad |f(x_j) - f(x_0)| \geq \epsilon_0 > 0.$$

Now, there are various cases to consider. Either $x_j < x_0$ or $x_j > x_0$. Similarly, either $f(x_j) \leq f(x_0) - \epsilon_0$ or $f(x_j) \geq f(x_0) + \epsilon_0$. One possibility is that there are infinitely many points x_j with $x_j < x_0$. If this is the case, we can renumber/rename the points and assume we have a sequence $\xi_j \nearrow x_0$. Then there are either infinitely many points ξ_j with $f(\xi_j) \leq f(x_0) - \epsilon_0$ or there are infinitely many points with $f(\xi_j) \geq f(x_0) - \epsilon_0$. Let's say the first thing happens, then we can rename points again and assume

$$-\frac{1}{j} < \xi_j < x_0 \qquad \text{and} \qquad f(\xi_j) \leq f(x_0) - \epsilon_0$$

for all $j = 1, 2, \ldots$. In this case, let b_0 be a point with $x_0 < b_0 < b$ and note that by convexity we must have for $j = 1, 2, 3, \ldots$

$$\begin{aligned} f(x_0) &= f\left(\left(1 - \frac{x_0 - \xi_j}{b_0 - \xi_j}\right)\xi_j + \frac{x_0 - \xi_j}{b_0 - \xi_j}\ b_0\right) \\ &\leq \left(1 - \frac{x_0 - \xi_j}{b_0 - \xi_j}\right) f(\xi_j) + \frac{x_0 - \xi_j}{b_0 - \xi_j}\ f(b_0) \\ &\leq \left(1 - \frac{x_0 - \xi_j}{b_0 - \xi_j}\right) [f(x_0) - \epsilon_0] + \frac{x_0 - \xi_j}{b_0 - \xi_j}\ f(b_0). \end{aligned}$$

Since this inequality holds for arbitrary j, we conclude

$$f(x_0) \leq \lim_{j \to \infty} \left[\left(1 - \frac{x_0 - \xi_j}{b_0 - \xi_j}\right) [f(x_0) - \epsilon_0] + \frac{x_0 - \xi_j}{b_0 - \xi_j}\ f(b_0)\right] = f(x_0) - \epsilon_0.$$

This is a contradiction.

Since the first possibility we considered led to a contradiction, we proceed to consider the other possibilities. It could be that we can find a sequence $\xi_1, \xi_2, \xi_3, \ldots$ satisfying

$$-\frac{1}{j} < \xi_j < x_0 \qquad \text{and} \qquad f(\xi_j) \geq f(x_0) + \epsilon_0$$

for all $j = 1, 2, \ldots$. In this case, we have by convexity for $j = 2, 3, \ldots$

$$\begin{aligned} f(\xi_j) &= f\left(\left(1 - \frac{\xi_j - \xi_1}{x_0 - \xi_1}\right)\xi_1 + \frac{\xi_j - \xi_1}{x_0 - \xi_1}\, x_0\right) \\ &\leq \left(1 - \frac{\xi_j - \xi_1}{x_0 - \xi_1}\right) f(\xi_1) + \frac{\xi_j - \xi_1}{x_0 - \xi_1}\, f(x_0). \end{aligned}$$

Notice that

$$\lim_{j\to\infty} \left(1 - \frac{\xi_j - \xi_1}{x_0 - \xi_1}\right) f(\xi_1) + \frac{\xi_j - \xi_1}{x_0 - \xi_1}\, f(x_0) = f(x_0).$$

In particular, for j large we conclude

$$f(\xi_j) < f(x_0) + \epsilon_0 \leq f(\xi_j).$$

The inequality $f(\xi_j) < f(\xi_j)$ is absurd. We have obtained another contradiction, and this means there cannot be infinitely many points x_j in the original sequence with $x_j < x_0$ and $|f(x_j) - f(x_0)| \geq \epsilon_0$. Thus, we must have infinitely many of the points x_j in this sequence with $x_0 < x_j$.

Renaming these points as above we can assume there is a sequence ξ_j satisfying

$$x_0 < \xi_j < \frac{1}{j}$$

and either

$$f(\xi_j) \leq f(x_0) - \epsilon_0 \qquad \text{or} \qquad f(\xi_j) \geq f(x_0) + \epsilon_0.$$

This situation again leads to two cases. In one of those case we have $f(\xi_j) \leq f(x_0) - \epsilon_0$ (for all $j = 1, 2, 3, \ldots$). If that happens, we can take a point a_0 with $a < a_0 < x_0$. Convexity then implies

$$\begin{aligned} f(x_0) &= f\left(\left(1 - \frac{x_0 - a_0}{\xi_j - a_0}\right) a_0 + \frac{x_0 - a_0}{\xi_j - a_0}\xi_j\right) \\ &\leq \left(1 - \frac{x_0 - a_0}{\xi_j - a_0}\right) f(a_0) + \frac{x_0 - a_0}{\xi_j - a_0} f(\xi_j) \\ &\leq \left(1 - \frac{x_0 - a_0}{\xi_j - a_0}\right) f(a_0) + \frac{x_0 - a_0}{\xi_j - a_0}[f(x_0) - \epsilon_0]. \end{aligned}$$

Since this inequality holds for all j, we can take the limit as j tends to infinity and conclude

$$f(x_0) \le f(x_0) - \epsilon_0.$$

This is again a contradiction. We are left with one final case.

In the final case $\xi_j > x_0$ and $f(\xi_j) \ge f(x_0) + \epsilon_0$ for all j. By convexity

$$\begin{aligned} f(\xi_j) &= f\left(\left(1 - \frac{\xi_j - x_0}{\xi_1 - x_0}\right) x_0 + \frac{\xi_j - x_0}{\xi_1 - x_0}\xi_1\right) \\ &\le \left(1 - \frac{\xi_j - x_0}{\xi_1 - x_0}\right) f(x_0) + \frac{\xi_j - x_0}{\xi_1 - x_0} f(\xi_1). \end{aligned}$$

For j large, the right side of the last inequality is strictly less than $f(x_0) + \epsilon_0$. We conclude that for j large $f(\xi_j) < f(\xi_j)$ and obtain a final contradiction.

The original sequence $x_1, x_2, x_3, \ldots$ existing in violation of the continuity of f at x_0 cannot exist. Therefore, f must be continuous at x_0, and since x_0 was taken as an arbitrary point in (a, b), the function f must be continuous on all of (a, b), that is $f \in C^0(a, b)$.

Problem 5 Use your example from Problem 2 above to illustrate the value of the **difference quotient**

$$\frac{f(x+h)-f(x)}{h}$$

with $x = 0$ and increment $h = -1$. (Hint: Start your illustration by drawing the graph of f.)

Solution: In Figure 7 I have labeled the positive lengths associated with

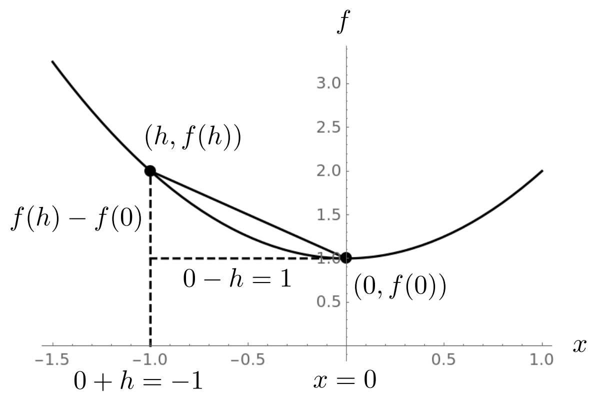

Figure 7: A difference quotient at a point $x = 0$ with a given increment $h = -1$ for a convex function.

$$|f(h) - f(0)| = f(h) - f(0)$$

and $|h| = -h = 1$. The actual difference quotient is

$$\frac{f(0+h)-f(0)}{h} = -\frac{f(h)-f(0)}{-h} < 0.$$

Problem 6 Show the derivative of the **absolute value function** $g : (-\infty, \infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} -x, & x \leq 0 \\ x, & x \geq 0 \end{cases}$$

is not well-defined at $x = 0$. Hint: Show the limit of the difference quotient does not exist as follows:

(a) Assume by way of contradiction that there exists a limit $L \in \mathbb{R}$ for which

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = L.$$

(b) Conclude there is some $\delta > 0$ for which

$$\left| \frac{|h|}{h} - L \right| < 1 \qquad \text{when} \qquad |h| < \delta.$$

(c) Get a contradiction by finding increments h_1 and h_2 satisfying $|h_j| < \delta$ for $j = 1, 2$ and

$$\left| \frac{g(0+h_2) - g(0)}{h_2} - \frac{g(0+h_1) - g(0)}{h_1} \right| \geq 2.$$

Hint hint: Use the triangle inequality.

Solution:

(a) I want to show the function g is **not** differentiable at $x = 0$. Thus, I will assume g is differentiable with a well-defined value $g'(0) = L$. The definition of differentiability then says

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = L.$$

(b) The existence of the limit in part **(a)** and the fact that the limit has value $L = g'(0)$ means that for any given $\epsilon > 0$ there exists some $\delta > 0$ depending on ϵ for which

$$\left| \frac{|h|}{h} - L \right| < \epsilon \qquad \text{when} \qquad |h| < \delta.$$

Simply taking the particular positive value $\epsilon = 1$, I can conclude there is some $\delta > 0$ for which

$$\left| \frac{|h|}{h} - L \right| < 1 \qquad \text{when} \qquad |h| < \delta.$$

(c) A particular choice of h with $|h| < \delta$ is $h_1 = -\delta/2$. Note that

$$g(h_1) = g(-\delta/2) = |-\delta/2| = \delta/2,$$

and the associated difference quotient is

$$\frac{g(h_1) - g(0)}{h} = \frac{\delta/2}{-\delta/2} = -1.$$

I conclude

$$\left|\frac{g(h_1) - g(0)}{h} - L\right| = |-1 - L| < 1.$$

Another valid choice is $h_2 = \delta/2$. In this case I conclude

$$\left|\frac{g(h_2) - g(0)}{h} - L\right| = |1 - L| < 1.$$

By the triangle inequality this means

$$\begin{aligned}
\left|\frac{g(h_2) - g(0)}{h_2} - \frac{g(h_1) - g(0)}{h_1}\right| &= \left|\frac{g(h_2) - g(0)}{h_2} - L + L - \frac{g(h_1) - g(0)}{h_1}\right| \\
&\leq \left|\frac{g(h_2) - g(0)}{h_2} - L\right| + \left|L - \frac{g(h_1) - g(0)}{h_1}\right| \\
&= \left|\frac{g(h_2) - g(0)}{h_2} - L\right| + \left|\frac{g(h_1) - g(0)}{h_1} - L\right| \\
&< 1 + 1 = 2.
\end{aligned}$$

On the other hand

$$\left|\frac{g(h_2) - g(0)}{h_2} - \frac{g(h_1) - g(0)}{h_1}\right| = |1 - (-1)| = 2.$$

We have thus shown $2 < 2$ which is absurd, and this shows conclusively that our assumption was incorrect. In this case, the assumption was that g was differentiable at $x = 0$. Since this is incorrect, we have a proof that g is not differentiable at $x = 0$ as desired.

Problem 7 Compute the **left derivative** $\delta : (-\infty, \infty) \to [-\infty, \infty]$ given by

$$\delta(x) = \lim_{h \nearrow 0} \frac{H(x+h) - H(x)}{h}$$

of the Heaviside function $H : (-\infty, \infty) \to \mathbb{R}$ by

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Note, δ is not a real valued function but rather an **extended real valued function** taking values in the **extended real line**, that is $\delta : (-\infty, \infty) \to \mathbb{R} \cup \{\pm\infty\}$.

Solution: For $x < 0$ every difference quotient

$$\frac{H(x+h) - H(x)}{h} = \frac{0 - 0}{h} \equiv 0,$$

for a small enough increment h, so the Heaviside function H is differetiable at these points with $H'(x) = 0$.

Similarly, for $x > 0$ difference quotients satisfy

$$\frac{H(x+h) - H(x)}{h} = \frac{1 - 1}{h} \equiv 0,$$

for a small enough increment h, so the Heaviside function H is differetiable at these points also with $H'(x) = 0$.

For the left derivative at $x = 0$ we consider

$$\frac{H(h) - H(0)}{h} = \frac{0 - 1}{h} = \frac{-1}{h} > 0$$

for $h < 0$. This quantity has a limit as $h \nearrow 0$ in the following sense: For any $N > 0$, there is some $\delta > 0$ such that $-\delta < h < 0$ implies

$$\frac{H(h) - H(0)}{h} = \frac{-1/h}{>}\frac{1}{\delta} > N.$$

(We just take $\delta < 1/N$.) Thus, we say

$$H'(0^-) = +\infty$$

and the left derivative of H is given by

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0. \end{cases}$$

Notice that the function δ is integrable with

$$\int_{\mathbb{R}} \delta = 0.$$

Problem 8 Recall (or note) the following two definitions: A function $f : (a,b) \to \mathbb{R}$ is **continuous at a point** $x \in (a,b)$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(\xi) - f(x)| < \epsilon \qquad \text{whenever} \qquad |\xi - x| < \delta.$$

A function $f : (a,b) \to \mathbb{R}$ is said to be **continuous on the interval** (a,b) if f is continuous at every point $x \in (a,b)$. In this case we write $f \in C^0(a,b)$. ($C^0(a,b)$ is the set of all real valued functions which are continuous on the interval (a,b).)

The Heaviside function h is continuous at every point $x \in (-\infty, 0) \cup (0, \infty)$, so we could write $h \in C^0((-\infty,0) \cup (0,\infty))$, but $h \notin C^0(\mathbb{R})$.

Draw the graph of $\sigma : (-\infty, \infty) \to \mathbb{R}$ by

$$\sigma(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and show $\sigma \in C^0(\mathbb{R})$.

Solution: If $x \neq 0$, then for $|\xi - x| < |x|/2$ we have $\xi x > 0$ and $|\xi| > |x|/2$ so $\xi x > x^2/2 > 0$ and ξ and x have the same sign in particular. With these preliminary inequalities, we can estimate:

$$\begin{aligned}
|\sigma(\xi) - \sigma(x)| &= \left| \xi \sin \frac{1}{\xi} - x \sin \frac{1}{x} \right| \\
&= \left| \xi \sin \frac{1}{\xi} - \xi \sin \frac{1}{x} + \xi \sin \frac{1}{x} - x \sin \frac{1}{x} \right| \\
&\leq |\xi| \left| \sin \frac{1}{\xi} - \sin \frac{1}{x} \right| + |\xi - x| \\
&\leq |x| \ |\cos t_*| \left| \frac{1}{\xi} - \frac{1}{x} \right| + |\xi - x| \\
&= |x| \ |\cos t_*| \frac{|\xi - x|}{\xi x} + |\xi - x| \\
&\leq \left(|x| \ \frac{2}{x^2} + 1 \right) |\xi - x|.
\end{aligned}$$

I have used the fact that the sine function is differentiable (with derivative given by the cosine function) and the mean value theorem so that t_* is some real number between $1/\xi$ and $1/x$. I've also used the estimate $|\cos t_*| \leq 1$. These can perhaps be considered elementary facts; it's a bit of a matter of taste, but in a certain sense the point of the problem is for you to learn how to make estimates like the ones I've made above. Of course, one could simply say sine is differentiable and the reciprocal function $\rho : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $\rho(x) = 1/x$ is differentiable, so the composition $\sigma = \sin \rho$ is differentiable and hence continuous on $\mathbb{R}\backslash\{0\}$. One can always quote theorems if one knows the theorems and knows how to quote/use them correctly, but if you learn to estimate, then sometimes you can figure out what to do when you do not have a theorem. That also might be the point. Again, it's just a matter of taste.

Notice that in the estimate I have obtained the number

$$\alpha = |x| \, \frac{2}{x^2} + 1 = \frac{2}{|x|} + 1$$

is a fixed positive number independent of ξ. Thus, for any $\epsilon > 0$, I may take

$$\delta = \frac{\epsilon}{2} \, \min\left\{ \frac{|x|}{2}, \frac{1}{\alpha} \right\} > 0.$$

I conclude furthermore that if $x \neq 0$, and $|\xi - x| < \delta$ for this choice of δ, then

$$|\sigma(\xi) - \sigma(x)| < \alpha\delta \leq \frac{\epsilon}{2} < \epsilon.$$

I conclude that σ is continuous at x.

It remains to consider the case $x = 0$. In this case, if $\xi = 0$ as well, then $|\sigma(\xi) - \sigma(x)| = 0$. If $\xi \neq 0$, then

$$|\sigma(\xi) - \sigma(x)| = \left| \xi \sin \frac{1}{\xi} \right| \leq |\xi|.$$

Thus, this is the easy case since for any $\epsilon > 0$, if we take $\delta = \epsilon > 0$, then we have

$$|\sigma(\xi) - \sigma(0)| \leq |\xi| < \epsilon \qquad \text{whenever} \qquad |\xi| < \delta.$$

I have shown σ is continuous at all points $x \in \mathbb{R}$ and thus $\sigma \in C^0(\mathbb{R})$.

It may be of some note that σ is not differentiable at $x = 0$, but $x\sigma$ is.

Problem 9 Recall the following three definitions: A function $f : (a, b) \to \mathbb{R}$ is **differentiable at a point** $x \in (a, b)$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists (as a finite real number). A function $f : (a, b) \to \mathbb{R}$ is said to be **differentiable on the interval** (a, b) if f is differentiable at every point $x \in (a, b)$. In this case $f' : (a, b) \to \mathbb{R}$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is a well-defined function.

Given a function $f : (a, b) \to \mathbb{R}$ which is differentiable on the interval (a, b), we say f is **continuously differentiable** if $f' \in C^0(a, b)$. In this case, we write $f \in C^1(a, b)$.

Find an example of a function $f : (a, b) \to \mathbb{R}$ which is differentiable on the interval (a, b), but is **not** continuously differentiable.

Solution: Indeed, expanding on what I suggested might be of note in the solution of the last problem, the function $\sigma \in C^0(\mathbb{R}) \backslash C^1(\mathbb{R})$. In fact, σ is not even differentiable at $x = 0$.

Taking a step back, the function

$$\tau(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

satisfies $\tau \notin C^0(\mathbb{R})$ because τ is not continuous at $x = 0$. Thus, one sees the regularity is increased by the factor of x.

Thus, consideration of the new function $\nu = x\sigma$ might be worthy of consideration. In fact,

$$\nu(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a function satisfying $\nu \in C^0(\mathbb{R}) \backslash C^1(\mathbb{R})$, but still this function ν might be differentiable. The function ν is of course continuous at all points (just use the same kinds of estimates from the previous problem) and ν is differentiable at all points x with $x \neq 0$. A difference quotient at $x = 0$ for ν takes the form

$$\frac{\nu(h) - \nu(0)}{h} = \frac{h^2 \sin(1/h)}{h} = \sigma(h).$$

Of course, we can assume $h \neq 0$ here because when we take the limit of a difference quotient, we assume $h \neq 0$ anyway.

We know the limit

$$\lim_{h \to 0} \frac{\nu(h) - \nu(0)}{h} = \lim_{h \to 0} \sigma(h) = \sigma(0) = 0$$

from the last problem. Thus, ν is differentiable with $\nu'(x)$ well-defined for each $x \in \mathbb{R}$. It remains to show the derivative $\nu' : \mathbb{R} \to \mathbb{R}$ given by

$$\nu'(x) = \begin{cases} 2x\sin(1/x) - 2\cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not continuous. Consider a sequence of points $x_1, x_2, x_3, \ldots$ with

$$x_j = \frac{1}{2\pi j} \qquad \text{for} \qquad j = 1, 2, 3, \ldots.$$

Since for any $\delta > 0$ one can take j large enough so that $|x_j| \leq 1/j < \delta$ (just take $j > 1/\delta$) and for every j we have

$$\nu'(x_j) = -2$$

it is clear that for $0 < \delta < 2$ and $j > 1/\delta$ there holds $|\nu'(x_j)| = |\nu'(x_j) - \nu'(0)| = 2$. Thus, ν' cannot be continuous at $x = 0$, and we know $\nu \in C^0(\mathbb{R}) \backslash C^1(\mathbb{R})$.

Problem 10 Recall (or note) the following two definitions: A function $f : (a, b) \to \mathbb{R}$ is **increasing** if

$$f(x_2) > f(x_1) \qquad \text{whenever} \qquad a < x_1 < x_2 < b.$$

A function $f : (a, b) \to \mathbb{R}$ is **decreasing** if

$$f(x_2) < f(x_1) \qquad \text{whenever} \qquad a < x_1 < x_2 < b.$$

A function which is increasing or decreasing is said to be (strictly) **monotone**.

Think about the function $f : (a, b) \to \mathbb{R}$ you suggested in Problem 1 above (proposed to be part of modeling a hanging slinky). We have a number of definitions in this assignment concerning continuity, differentiability, monotonicity, and convexity. If your function f turns out to have values which may be reasonably compared to measurements taken from the hanging slinky, what properties do you expect the model measurement function to have (in terms of continuity, differentiability, monotonicity, and convexity)? Also include a discussion of the expected **boundary values** $f(a)$ and $f(b)$. Tell me anything you think should be true about a reasonable model function, i.e., your model function, which you can assert (or think you can assert) without actually making any measurements.

Note: Please give careful attention to Problems 1 and 10 on this assignment. The topics addressed in these problems will come up again.

Solution of Problem 10: The stretch function $\sigma : [0, h_0] \to [0, \infty)$ of category **III** may be assumed to satisfy $\sigma(0) = 0$ and $\sigma(h) > 0$ for $0 < h \le h_0]$ with $\sigma(h_0)$ the full length of the hanging slinky which can clearly be measured to within some accuracy.

Hopefully, it's pretty obvious that σ should be monotone increasing and one-to-one (or injective) with range $[0, \sigma(h_0)]$ so that

$$\sigma : [0, h_0] \to [0, \sigma(h_0)].$$

I think it is also very natural to assume σ is (or should be) differentiable at least initially, and probably C^∞, though both of these assumptions definitely come into question if one considers carefully the relations of this particular stretch function with the more accurate/precise stretch functions of categories **I** and **II**. Setting this consideration aside, an initial expectation is

$$\sigma'(h) > 0 \qquad \text{for all } h \text{ with } 0 \le h \le h_0.$$

In fact, if we were to imagine the slinky were simply turned upside down and suspended from one end in zero gravity (and ideally remained entirely compressed), then we would have $\sigma(h) = h$ and $\sigma'(h) \equiv 1$. This suggests the further expectation

$$\sigma'(h) \geq 1.$$

If you think about this a bit more, the (observed) "density of the coils" decreases along with the value of σ, and this means we should expect $\sigma'(h)$ to decrease with a maximum at $h = 0$ and a minimum

$$\sigma'(h_0) \approx 1.$$

The heuristic reason for this is that the distance the coils corresponding to the value h are pulled apart is greater at points for which there is greater mass of the slinky below the point $X(h)$ in the extension. The farther down the hanging slinky one looks, the less slinky there is below a given point and hence less mass below the point. To make this more quantitative, if $0 < h_1 < h_2 < h_0$, then by the monotonicity we should expect $\sigma(h_1) < \sigma(h_2)$ and this means that if $X(h_j)$ is the location on the hanging slinky corresponding to h_j for $j = 1, 2$, then $X(h_2)$ is lower than $X(h_1)$ and there is more mass below $X(h_1)$ than there is below $X(h_2)$. Thus, we should expect

$$\sigma'(h_1) > \sigma'(h_2). \tag{3}$$

We will want to incorporate this kind of observation into our modeling to get an exact formula, or something like an exact formula—at least a differential equation, for σ later.

The monotonicity suggested by (3) tells one something about convexity. This says that the first derivative is decreasing, or with adequate regularity (and nominally we expect to have that) that the second derivative is negative. We have observed above that convexity is associated with a positive second derivative. In this case, we expect $-\sigma$ satisfies

$$-\sigma''(h) > 0.$$

That is, $-\sigma$ should be convex. Sometimes one says in this case that σ is "concave" or that σ is "convex down" or "concave down," but none of this terminology is particularly standard. The most standard thing one can say that most people will understand is "I expect $-\sigma$ should be convex."

I think I've pretty much covered what was asked in this problem. One boundary value is more or less definite: $\sigma(0) = 0$. The other one $\sigma(h_0)$ which models the length of the entire hanging slinky will need to be determined by measurement or modeling. At this stage, $\sigma(h_0)$ is an unknown but perhaps very interesting quantity.