Assignment 14: partial differential equations
Problem 10 Solution

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Problem 1 (the real gamma function) Consider $\Gamma : (0, \infty) \to \mathbb{R}$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) Compute $\Gamma(1)$.

(b) Show $\Gamma(x + 1) = x\Gamma(x)$ for $x > 0$.

(c) Show $\Gamma(n) = (n - 1)!$ for $n = 1, 2, 3, \ldots$.

(d) Show $\Gamma(1/2) = \sqrt{\pi}$.

(e) Show

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \qquad \text{for} \qquad n = 1, 2, 3, \ldots.$$

(f) Show the n-dimensional measure of an n-dimensional ball of radius r in $\mathbb{R}^n$ is $\omega_n r^n$ where

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$

(g) Show the $(n - 1)$-dimensional measure of $\partial B_r(\mathbf{p})$ is $n\omega_n$.

(h) Specialize the formula from part **(f)** to the special cases $n = 2k$ is even and $n = 2k + 1$ is odd to show ω_n is always a rational multiple of a power of π.

Problem 2 (mean value property) Show that if $u \in C^2(U)$ is harmonic on an open set $U \subset \mathbb{R}^n$ and $B_r(\mathbf{p}) \subset\subset U$, then

$$u(\mathbf{p}) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{p})} u.$$

Hint(s): Change variables in the integral so that you integrate over a domain $B_1(\mathbf{0})$ independent of r. Differentiate the expression you get with respect to r, and use the divergence theorem to show the average value is constant. Determine the constant value must be $u(\mathbf{p})$ (by continuity).

Problem 3 (heat equation) Find a Fourier sine series/separated variables solution of the heat evolution problem

$$\begin{cases} u_t = \Delta u, & (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = 0 = u(\pi,t), & t > 0 \\ u(x,0) = \pi/2 - |x - \pi/2|, & 0 \le x \le \pi. \end{cases}$$

Problem 4 (heat equation) Use mathematical software to illustrate the solution you found in Problem 3 above. What is interesting about the (apparent) regularity of the solution?

heat equation with insulated boundary conditions

For Problems 5-7 consider the problem

$$\begin{cases} u_t = \Delta u, & \text{on } (0,\pi)\times(0,\infty) \\ u(0,t) = 0, & t > 0 \\ u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = x, & 0 \le x \le \pi \end{cases} \tag{1}$$

Problem 5 (heat equation with insulated boundary conditions) Find the **steady state** temperature distribution $u_0(x)$ for (1).

Problem 6 (heat equation with insulated boundary conditions) Find a Fourier series/separated variables solution of (1). You will need to find and solve the appropriate Sturm-Liouville problem; you can't just use a sine series.

Problem 7 (heat equation with insulated boundary conditions) Use mathematical software to illustrate the solution you found in Problem 6 above.

One-dimensional wave equation on a finite interval

For problems 8 and 9 we consider the following initial/boundary value problem

$$\begin{cases} u_{tt} = u_{xx} & \text{for } (x,t) \in (0,2) \times [0,\infty) \\ u(x,0) = x + 1/2 - |x-1|/2, & x \in [0,2] \\ u_t(x,0) = 0, & x \in [0,2] \\ u(0,t) = 0, & t \geq 0 \\ u(2,t) = 2, & t \geq 0 \end{cases} \tag{2}$$

which is assumed to model the logitudinal deformation of a one-dimensional elastic continuum. I suggest you illustrate the model function u by representing/plotting a sequence of twenty-one representative parameter/material points $x_j = j/10$ for $j = 0, 1, 2, \ldots, 20$ as follows: The equilibrium configuration for the elastic continuum is represented by the spatial identity $u_0(x,t) \equiv x$ as indicated in Figure 1. With

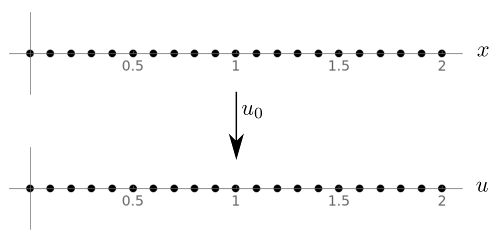

Figure 1: The identity deformation of a one-dimensional continuum

this approach, the initial displacement $u(x,0) = x + 1 - |x-1|$ can be illustrated as indicated in Figure 2 with $u(1,0) = 3/2$.

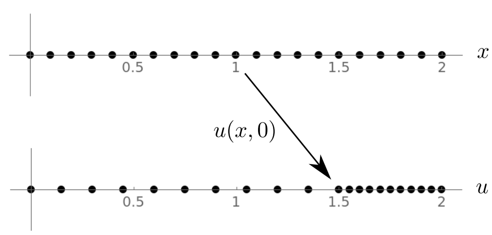

Figure 2: Initial deformation of a one-dimensional continuum

Problem 8 (wave equation; Fourier series solution; Boas Chapter 13 Section 4) Let $w(x,t) = u(x,t) - x$ where u is the solution of (2), and solve the initial/boundary value problem satisfied by w with w given as a superposition of separated variables solutions.

Problem 9 (wave equation) Animate the function $u(x,t)$ obtained in the previous problem using the mapping approach illustrated in Figures 1 and 2 above (with time t as the animation parameter).

Problem 10 (integral identities for the multi-dimensional wave equation) Consider the initial/boundary value problem

$$\begin{cases} u_{tt} = \Delta u & \text{on } U \times (0,\infty) \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in U \\ u_t(\mathbf{x},0) = v_0(\mathbf{x}), & \mathbf{x} \in U \\ u(\mathbf{x},t) = \phi(\mathbf{x},t), & \text{on } \partial U \times [0,\infty) \end{cases} \tag{3}$$

where U is a bounded domain with C^1 boundary in $\mathbb{R}^n$ and u_0, v_0, and ϕ are given smooth functions. Assume $u \in C^2(\overline{U} \times [0,\infty))$ is a solution of (3) and consider the "energy" quantity

$$E(t) = \frac{1}{2} \int_U \left[u_t^2 + |Du|^2 \right] \tag{4}$$

which may be considered as a sum of kinetic and potential energies.

(a) Calculate the derivative

$$\frac{dE}{dt}$$

and use the divergence theorem to express this quantity in terms of the boundary values.

solution:

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\int_U \frac{\partial}{\partial t}\left[u_t^2 + Du\cdot Du\right] \\
&= \int_U \left[u_t u_{tt} + Du\cdot (Du)_t\right] \\
&= \int_U \left[u_t \Delta u + Du\cdot Du_t\right] \\
&= \int_U \left[\operatorname{div}(u_t Du) - Du_t\cdot Du + Du\cdot Du_t\right] \\
&= \int_U \operatorname{div}(u_t Du) \\
&= \int_{\partial U} u_t Du\cdot n \\
&= \int_{\partial U} u_t D_n u \\
&= \int_{\partial U} \phi_t D_n \phi.
\end{aligned}
$$

(b) Give conditions on the function ϕ under which the energy is conserved, i.e.,

$$\frac{dE}{dt} \equiv 0.$$

solution: There are two obvious conditions:

(i) The time derivative of ϕ restricted to the boundary vanishes identically:

$$\left[\phi_t(\mathbf{x}, t)\right]\Big|_{\mathbf{x}\in\partial U} \equiv 0,$$

(ii) The normal derivative of ϕ restricted to the boundary vanishes identically:

$$\left[D_n\phi(\mathbf{x}, t)\right]\Big|_{\mathbf{x}\in\partial U} \equiv 0.$$

Of course, these two may be combined by saying there is a (measurable) decomposition of the boundary $\partial U = A \cup B$ for which

$$\left[\phi_t(\mathbf{x}, t)\right]\Big|_{\mathbf{x}\in A} \equiv 0,$$

and

$$[D_n \phi(\mathbf{x}, t)]\Big|_{\mathbf{x} \in B} \equiv 0.$$

(c) Interpret the conditions you gave in part **(b)** above in terms of physical model assumptions for a two-dimensional ($n = 2$) elastic membrane.

solution: Recalling the meaning of the boundary condition $u = \phi$, we see that at points where $\phi_t(\mathbf{x}, t) = u_t(\mathbf{x}, t) = 0$ one is requiring

the boundary position is held constant in time.

This interpretation may apply to the entire boundary as in condition **(i)** or to only a portion $A \subset \partial U$.

At points where $D_n \phi(\mathbf{x}, t) = Du(\mathbf{x}, t) \cdot n = 0$, one is requiring

the normal derivative of the solution
vanishes at the boundary

that is for a two-dimensional membrane

the membrane is clamped
in a horizontal position at the boundary

at such points, though the actual displacement can vary with time.

It is intuitively quite clear that an expression for total energy, e.g., the quantity E, might remain constant when the boundary displacement is fixed. It is (to me) rather less obvious why a total energy should be conserved in a situation where the boundary is moved up and down but clamped so as to remain horizontal.

There could have been a part **(d)** to this problem in which one is asked to show uniqueness for the initial/boundary value problem. In that case, the difference $w = u - v$ of two solutions would vanish along with its time derivative w_t at $t = 0$ giving

$$E_w(0) = \int_U \left[w_t^2 + |Dw|^2 \right] = 0.$$

And, furthermore the boundary distribution $w(\mathbf{x}, t)$ for $\mathbf{x} \in \partial U$ would vanish identically, so that by part **(b)** condition **(i)** we have

$$\frac{dE_w}{dt} \equiv 0.$$

Therefore, $E_w \equiv 0$, and $|Dw| \equiv 0$ in particular. This means w is a constant, and specializing back to the initial values tells us the constant must be zero. That is, $u \equiv v$.

Bonus Problems

Problem 11 (calculus of variations) Formulate/model the energy (potential energy due to gravity) associated with a symmetric hanging chain given as the graph of a function $u \in C^1[-1,1]$ with $u(-1) = 0 = u(1)$ and length $L = 4$.

Problem 12 (calculus of variations) Use the method of Lagrange multipliers to find an ODE satisfied by the model function u of the previous problem:

(a) Let

$$\mathcal{B} = \left\{ u \in C^1[-1,1] : u(-1) = 0 = u(1), \text{ and } \int_{-1}^{1} \sqrt{1 + u'(x)^2}\, dx = 4 \right\}$$

Consider $\mathcal{F}[u] = \mathcal{E}[u] - \lambda \operatorname{length}[u]$ where $\mathcal{E}$ is the potential energy. Show that if $u_0 \in C^2[-1,1] \cap \mathcal{B}$ satisfies

$$\mathcal{E}[u_0] \le \mathcal{E}[u] \qquad \text{for all } u \in \mathcal{B},$$

then there exists some $\lambda \in \mathbb{R}$ such that $\mathcal{F}[u_0] \le \mathcal{F}[u]$ for all

$$u \in \mathcal{A} = \{ w \in C^1[-1,1] : w(-1) = 0 = w(1) \}.$$

(b) Compute the first variation $\delta \mathcal{F}_{u_0}[\phi]$ for $\phi \in C_c^\infty[-1,1]$, and use the fundamental lemma of the calculus of variations to find all C^2 minimizers u_0 of $\mathcal{F}$.

(c) Solve the ODE from part **(b)** above, and numerically find λ to find the model shape of the hanging chain. Hint:

$$\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Problem 13 Use the potential energy due to gravity along with the elasticity model for the tension force in the hanging slinky to model the hanging slinky using the calculus of variations.