Assignment 13:
Integration and Laplace's equation
Due Wednesday, April 22, 2023

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Problem 1 (Laplace's equation and Poisson's equation) Assume $g \in C^2(\partial B_1(0,0))$ where $U = B_1(0,0)$ is the open unit disk in $\mathbb{R}^2$ and the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{on } U \\ u\big|_{\partial U} = g \end{cases} \tag{1}$$

for Laplace's equation.

(a) Recall problems 2-7 of Assignment 5. Formulate and state precisely a definition of what the condition $g \in C^2(\partial B_1(0,0))$ above means.

(b) Find an extension $G \in C^2(\overline{B_1(0,0)})$ on the entire closed unit disk with restriction to $\partial B_1(0,0)$ given by g.

(c) Formulate a Dirichlet problem for Poisson's equation of the form

$$\begin{cases} \Delta v = f, & \text{on } U \\ v\big|_{\partial U} = 0 \end{cases} \tag{2}$$

and show that when (and if) you can solve this problem, you can solve (1). Hint(s): What you need to do first here is specify the function $f : U \to \mathbb{R}$. Then you need to think about what it means to solve (2) and (1). Maybe you should not take my suggested order too seriously.

Problem 2 (Dirichlet problem for Laplacian; flux) Let U be a bounded open set in $\mathbb{R}^n$ with C^1 boundary. Assume $u \in C^2(U) \cap C^1(\overline{U})$ is a solution of the Dirichlet problem (1) for Laplace's equation. Show

$$\int_{\partial U} Du \cdot n = 0$$

where n is the outward unit normal to ∂U. Hint: Integrate the PDE.

Problem 3 (uniqueness for Poisson's equation and Laplace's equation) Let U be a bounded open set in $\mathbb{R}^n$ with C^1 boundary. Assume $u, v \in C^2(U) \cap C^1(\overline{U})$ be solutions of the Dirichlet problem

$$\begin{cases} \Delta u = f, & \text{on } U \\ u\Big|_{\partial U} = g. \end{cases} \tag{3}$$

(a) Derive the product rule

$$\operatorname{div}(\phi \mathbf{v}) = D\phi \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v}$$

for the divergence of a scalar function times a field $\mathbf{v}$.

(b) Apply the product rule for the divergence to the field uDu.

(c) Prove $u \equiv v$, i.e., the solutions of (3) are unique. Hint: Apply part the identity of **(b)** to to the field $(u - v)D(u - v)$.

Problem 4 (Laplace's equation; derivation) Derive Laplace's equation for a laminar region $U \subset \mathbb{R}^2$ using the law of specific heat and Fourier's law of heat conduction. To do a thorough job, you should define all quantities involved, give the units/physical dimensions of those quantities, and state precisely all physical (model) laws.

Fourier series solution of Laplace's equation

For problems 5-8 let $U = (0,2) \times (0,1)$ be an open rectangle in $\mathbb{R}^2$ with boundary component $\Gamma = \{(x,1) : x \in (0,2)\}$. Consider the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{on } U \\ u(x,1) = e - e^{(x-1)^2}, & x \in (0,2) \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial U \backslash \Gamma \end{cases} \tag{4}$$

Problem 5 (Laplace's equation; Fourier series solution part 1) Find all separated variables solutions $v(x,y) = A(x)B(y)$ of the incomplete boundary value problem

$$\begin{cases} \Delta u = 0, & \text{on } U \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial U \backslash \Gamma \end{cases} \tag{5}$$

Problem 6 (Laplace's equation; Fourier series solution part 2) You should have found a sequence $\{v_j\}_{j=1}^{\infty}$ of separated variables solutions in Problem 5 with $A_j(x)/\sin(j\pi x)$ constant for $x \in (0,2)$ and such that $cv_j \in C^\infty(\mathbb{R}^2)$ is a solution of (5) for each $c \in \mathbb{R}$. Find a solution of (4) having the form

$$u(x,y) = \sum_{j=1}^{\infty} c_j v_j(x,y).$$

You should find the coefficients c_j. The expressions you get may involve integrals you cannot evaluate in closed form.

Problem 7 (Laplace's equation; Fourier series solution part 3) You should have found a Fourier sine expansion for the function $g(x) = u(x,1)$ in Problem 6.

(a) Use mathematical software, e.g., Matlab, Mathematica, Maple, to obtain numerical approximations for the first three nonzero coefficients, and the integrals involved in particular, in your Fourier series.

(b) Plot the function g along with a partial sum g_j of the Fourier series including j (nontrivial) terms for $j = 1, 2, 3$.

(c) Plot the differences $\delta_j = g_j - g$ for $j = 1, 2, 3$.

(d) Determine how many terms are required in your Fourier expansion to get a partial sum g_j satisfying

$$|g_j(x) - g(x)| < \delta = 0.00001 \qquad \text{for} \qquad 0 \le x \le 2.$$

Problem 8 (Laplace's equation; Fourier series solution part 4) You should have found a partial sum g_j in your Fourier sine expansion of $g(x) = u(x, 1)$ in Problem 7 part **(d)** for which g_j approximates $u(x, y)$ along Γ uniformly with tolerance $\delta = 0.0001$. Use the weak maximum principle (Assignment 12 Problem 6) to show the corresponding partial sum for your solution u approximates the actual solution u uniformly on U to within the same tolerance.

Problem 9 (divergence theorem; Boas Problem 6.10.5) Let $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 25\}$ and let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{v}(\mathbf{p}) = |\mathbf{p}|^2\mathbf{p}$. Compute

$$\int_V \operatorname{div} \mathbf{v}.$$

Problem 10 (Laplace's equation; uniqueness of solutions) Let $U = B_1(0, 0) \subset \mathbb{R}^2$ be the unit disk and consider the mixed boundary value problem

$$\begin{cases} \Delta u = 0, & \text{on } U \\ u(\mathbf{e}_1) = 5, & \\ (D_n u)\Big|_{\partial U \backslash \{\mathbf{e}_1\}} = 3. & \end{cases} \tag{6}$$

Assume $u, v \in C^2(U) \cap C^1(\overline{U})$ are solutions of (6).

(a) Apply the divergence theorem to $\operatorname{div}[(u - v)D(u - v)]$ to conclude $u - v = c$ is constant on U.

(b) Show carefully that the constant c must be zero.

(c) Use what you know about the Laplace operator, solutions of Laplace's equation etc. to sketch what you think the graph of a solution of this problem must look like.

(d) Do you think this problem has a solution in $C^2(U) \cap C^1(\overline{U})$?