Math 6702, Assignment 12

Mean Value Properties of Solutions of Laplace's Equation on $\mathbb{R}^2$

In the first four problems below U is an open bounded subset of $\mathbb{R}^2$.

1. Let $u \in C^2(U)$ be a classical solution of $\Delta u = 0$ on U. Show the following:

$$\text{If } B_r(\mathbf{x}_0) \subset\subset U, \text{ then} \quad u(\mathbf{x}_0) = \frac{1}{2\pi r}\int_{\partial B_r(\mathbf{x}_0)} u = \frac{1}{2\pi r}\int_{\mathbf{x}\in\partial B_r(\mathbf{x}_0)} u(\mathbf{x}). \tag{1}$$

   Notice that the expression on the right is the average value of u over $\partial B_r(\mathbf{x}_0)$.

   Hint(s): Set

$$f(r) = \frac{1}{2\pi r}\int_{\mathbf{x}\in\partial B_r(\mathbf{x}_0)} u(\mathbf{x}),$$

   and compute $f'(r)$. Then consider the limit of $f(r)$ as $r \searrow 0$.

2. Let $u \in C^2(U) \in C^0(\overline{U})$ be a classical solution of $\Delta u = 0$ on U. Show the following:

$$\text{If } u\Big|_{\partial U} \geq 0, \quad \text{then} \quad u \geq 0 \text{ on } U.$$

3. Let $u \in C^1(U)$ be a weak solution of $\Delta u = 0$ on U. Show (1) still holds. Hint(s): Extend u to $\bar{u} = u\chi_{B_{r+\epsilon}(\mathbf{x}_0)}$ where $B_{r+\epsilon}(\mathbf{x}_0) \subset\subset U$. Mollify $\bar{u}$ and show $\mu \star \bar{u}$ converges uniformly to u as the mollifier μ intensifies. Finally, show that for intense mollification, $v = \mu \star \bar{u}$ is a classical solution of $\Delta v = 0$ on $B_{r+\epsilon/2}(\mathbf{x}_0)$.

4. Let $u \in C^1(U)$ be a weak solution of $\Delta u = 0$ on U satisfying

$$u\Big|_{\partial U} \geq 0.$$

   Assume U is connected. Assume there is another (nonempty) open set $U_0 \subset U$ with

$$u\Big|_{U_0} \equiv 0.$$

   Show $u \equiv 0$. Hint(s): Assume (BWOC) $U_0 \subset \{\mathbf{x} \in U : u(\mathbf{x}) = 0\} \neq U$. Find, i.e., show there exists, a ball $B_r(\mathbf{x}_0) \subset\subset U$ with

$$B_r(\mathbf{x}_0) \subset U_0 \qquad \text{but} \qquad B_{r+\epsilon}(\mathbf{x}_0) \not\subset U_0 \quad \text{for all } \epsilon > 0.$$

   Use the mean value property to get a contradiction.

Just so you know and in case you want to read about it, there are much stronger results than those given in problems 2 and 4 above, but the proof is more difficult and needs more than just the mean value property/formula of problem 1. Specifically, for problem 2 one can prove that given a connected component U_* of U either $u > 0$ on U_* or $u \equiv 0$ on U_*. This amounts to what is called the **E. Hopf strong maximum principle**. The usual way to prove it is with a result called the **E. Hopf boundary point lemma**. Incidentally, Eberhard Hopf is not the mathematician you have (do doubt) heard of in connection with the famous Hopf invariant and Hopf fibration. That would be Heinz Hopf.

Green's Theorem §6.9 (Boas)

5. You know Gauss' theorem (or the divergence theorem) in the plane which says that given a bounded C^1 open domain $U \subset \mathbb{R}^2$ in the domain of a vector field $\mathbf{v}$ we have

$$\int_U \operatorname{div} \mathbf{v} = \int_{\partial U} \mathbf{v} \cdot \mathbf{n}.$$

Use Gauss' theorem to prove Green's theorem:

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{\partial U} \mathbf{v} \cdot T$$

where $\mathbf{v} = (P, Q)$ and T is the counterclockwise unit normal around ∂U.

Stokes' Theorem §6.11 (Boas)

Stokes' theorem states that if $\mathcal{S}$ is an oriented surface in $\mathbb{R}^3$ in the domain of a (differentiable) vector field $\mathbf{v}$ and having C^1 boundary $\partial\mathcal{S}$, then

$$\int_{\mathcal{S}} \operatorname{curl} \mathbf{v} \cdot N = \int_{\partial\mathcal{S}} \mathbf{v} \cdot T$$

where N is the unit normal orienting $\mathcal{S}$ and T is the counterclockwise unit tangent around $\partial\mathcal{S}$ with respect to N.

The following is not the most wonderful problem in the world, but it is kind of fun. If you've been following/picking up on what I've been saying about integration this semester, then it should be way too easy...even sort of juvenile. If it's not like this, then start back with the basics of integration and become an integration ninja.

6. (6.11.16) According to Maxwell's equations (in the potential formulation) any **magnetic field** $B : U \to \mathbb{R}^3$ where U is a simply connected domain in $\mathbb{R}^3$ satisfies

$$\operatorname{div} B = 0 \qquad \text{and} \qquad B = \operatorname{curl} A$$

where A is the **magnetic vector potential**. Observe that

$$\begin{aligned} 0 &= \int_U \operatorname{div} B \\ &= \int_{\mathcal{S}} B \cdot N \qquad \text{where } \mathcal{S} = \partial U \text{ by the divergence theorem} \\ &= \int_{\mathcal{S}} \operatorname{curl} A \cdot N \\ &= \int_{\partial\mathcal{S}} A \cdot T \qquad \text{by Stokes theorem.} \end{aligned}$$

If for every closed loop $\Lambda = \partial\mathcal{S}$ we have

$$\int_\Lambda A \cdot T = 0,$$

then A is conservative. Therefore, there exists a potential function ψ with $A = D\psi$. Consequently,

$$B = \operatorname{curl} A = \operatorname{curl} D\psi = \mathbf{0},$$

so **all magnetic fields are zero fields**. (You can check by calculation that it's always true that the curl of a gradient always vanishes.) Find the error(s) in this lovely "proof." Incidentally, the divergence of a curl always vanishes too. We don't use that here, but it's good to know.

I think we've pretty much covered (at some level) Chapter 4 (differentiation), Chapter 5 (integration), and Chapter 6 (vector analysis) of Boas. It would have been nice to go over Chapter 13 (PDE) in more detail, but I think with what we did do, there's nothing in Chapter 13 you can't read easily. If you read a couple pages of Boas from time to time, she'll keep you sharp on your applied math, so it's a good book to know about/have.