Assignment 12:
Integration and the heat equation
Due Wednesday, April 17, 2023

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Problem 1 (The divergence of a vector field) Let $\mathbf{v} : U \to \mathbb{R}^2$ be a vector field with component functions $\mathbf{v} = (v_1, v_2)$ satisfying $v_j \in C^1(U)$. Recall that the **divergence** of $\mathbf{v}$ is defined by

$$\operatorname{div} \mathbf{v}(\mathbf{p}) = \lim_{Q \to \{\mathbf{p}\}} \frac{1}{\operatorname{area}(Q)} \int_{\partial Q} \mathbf{v} \cdot \nu$$

where $\mathbf{p} \in U$ and ν is the outward unit conormal to ∂Q. Take

$$Q = Q_\epsilon(\mathbf{p}) = \left\{ \mathbf{x} \in \mathbb{R}^2 : |x_j - p_j| < \frac{\epsilon}{2} \right\}$$

and compute

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon^2} \int_{\partial Q} \mathbf{v} \cdot \nu$$

in terms of the first partial derivatives of v_1 and v_2.

Problem 2 (heat equation; average temperature) Let $u \in C^2(\overline{U} \times [0, \infty))$ be a solution of the heat equation

$$u_t = \Delta u.$$

Calculate the time rate of change of the spatial average temperature

$$\frac{1}{\mu(U)} \int_U u.$$

Use the divergence theorem to express your answer in terms of the flux integral

$$\int_{\partial U} D_n u$$

of the gradient.

Problem 3 (heat equation; L^2 norm) Let $u \in C^2(\overline{U} \times [0, \infty))$ be a solution of the heat equation

$$u_t = \Delta u.$$

(a) Calculate the time rate of change of the quantity

$$\|u\|_{L^2}^2 = \int_U u^2.$$

Use the divergence theorem to express your answer in terms of the L^2 norm of the gradient

$$\int_U \|Du\|^2.$$

Hint: $\operatorname{div}(uDu) = Du \cdot Du + u \operatorname{div} Du$.

(b) Use Fourier's law (of heat conduction) to determine a condition to model a **perfectly insulated boundary** during the evolution of u. Hint: There is zero heat flux across the boundary.

(c) Assuming a perfectly insulated boundary condition, show the spatial L^2 norm of u defined in part **(a)** above is non-increasing as a function of time.

Problem 4 (heat equation; uniqueness) Let $u, v \in C^2(\overline{U} \times [0, \infty))$ be solutions of the (same) heat evolution initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & \text{on } U \times [0, \infty) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in U \\ u(\mathbf{x}, t) = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \partial U \times [0, \infty). \end{cases}$$

Show $u \equiv v$. Hint: Compute

$$\frac{d}{dt} \int_U (u - v)^2.$$

This is like the computation in Problem 2 above with $h = u - v$ in place of u.

Problem 5 (heat equation; uniqueness) Let U be an open domain in $\mathbb{R}^n$ with C^1 boundary, and let $u, v \in C^2(\overline{U} \times [0, \infty))$ be solutions of the (same) heat evolution initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & \text{on } U \times [0,\infty) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in U \\ Du(\mathbf{x}, t) \cdot n = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \partial U \times [0, \infty) \end{cases}$$

Here n denotes the outward unit conormal along ∂U. Show $u \equiv v$.

Problem 6 (Laplace's equation; maximum principle) Let U be a bounded domain in $\mathbb{R}^2$ with C^1 boundary and let $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic in U.

(a) Given any point $\mathbf{p} \in U$, any value $v_0 > \max_{\partial U} u$, and any $\epsilon > 0$, compute Δh where $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(\mathbf{x}) = v_0 - \epsilon \|\mathbf{x} - \mathbf{p}\|^2.$$

(b) Show that if $u(\mathbf{x}) \leq h(\mathbf{x})$ for $\mathbf{x} \in U$ where h is the function from part **(b)** above, then $u(\mathbf{x}) < h(\mathbf{x})$ for $\mathbf{x} \in U$. Hint: Assume equality $u(\mathbf{q}) = h(\mathbf{q})$ holds for some $\mathbf{q} \in U$, and show $\Delta u(\mathbf{q}) \leq \Delta h(\mathbf{q})$.

(c) Show

$$u(\mathbf{x}) \leq \max_{\partial U} u \qquad \text{for every } \mathbf{x} \in U. \tag{1}$$

This is called the **weak maximum principle**. The **strong maximum principle** asserts that the inequality in (1) is strict.

Problem 7 (integration; Boas 5.4.20) Consider the domain

$$U = \left\{ (x, y) \in \mathbb{R}^2 : x < y < 1 - x,\ 0 < x < \frac{1}{2} \right\}.$$

(a) Find a domain $V \subset \mathbb{R}^2$ for which $\Psi : V \to U$ is one-to-one and onto with

$$\Psi(r, s) = \left(\frac{r - s}{2}, \frac{r + s}{2} \right).$$

(b) Calculate

$$\int_U f$$

where $f : U \to \mathbb{R}$ by

$$f(x,y) = \left(\frac{x-y}{x+y}\right)^2.$$

Hint: Change variables.

Problem 8 (parameterization; Boas 5.4.22) A wire is coiled around a circular cylindrical rod having radius 1 inch and length 1 foot. If there are three turns of wire per inch along the length, what is the length of the wire?

Problem 9 (divergence in cylindrical coordinates) For this problem, let $\Psi : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ by

$$\Psi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

denote the cylindrical coordinates map. Consider the set

$$Q_{\epsilon,\delta,\zeta}(\mathbf{p}) = \{(r\cos\theta, r\sin\theta, z) : |r-r_0| < \epsilon,\ |\theta-\theta_0| < \delta,\ |z-z_0| < \zeta\}$$

where $r_0 > 0$ and θ_0 and z_0 are fixed and $0 < \epsilon < r_0$ and δ and ζ are fixed and positive. Let's call this a **cylindrical cube**. Given a vector field $\mathbf{v} : U \to \mathbb{R}^3$ with C^1 component functions and U an open subset of $\mathbb{R}^3$ with $\mathbf{p} = (r_0\cos\theta_0, r_0\sin\theta_0, z_0) \in U$, note that the functions

$$\begin{aligned}
\rho(r,\theta,z) &= \mathbf{v}(r\cos\theta, r\sin\theta, z)\cdot(\cos\theta,\sin\theta,0),\\
\phi(r,\theta,z) &= \mathbf{v}(r\cos\theta, r\sin\theta, z)\cdot(-\sin\theta,\cos\theta,0),\ \text{and}\\
h(r,\theta,z) &= \mathbf{v}(r\cos\theta, r\sin\theta, z)\cdot\mathbf{e}_3
\end{aligned}$$

are well-defined and C^1 in some open neighborhood of (r_0,θ_0,z_0). Compute the divergence

$$\operatorname{div}\mathbf{v}(\mathbf{p}) = \lim_{\epsilon,\delta,\zeta\to 0}\frac{1}{\operatorname{vol}(Q_{\epsilon,\delta,\zeta}(\mathbf{p}))}\int_{Q_{\epsilon,\delta,\zeta}(\mathbf{p})}\mathbf{v}\cdot N$$

in terms of the first partial derivatives of ρ, ϕ, and h.

Problem 10 (flow across a line) If $\rho\mathbf{v}$ is a mass density flow field in $\mathbb{R}^2$, i.e.,

$$[\rho\mathbf{v}] = \frac{M}{LT} \qquad \text{and} \qquad [\rho] = \frac{M}{L^2},$$

and $\mathbf{v} \equiv (\cos\theta, \sin\theta)$ is constant, then how much mass flows across the y-axis in each unit of time?