Math 6702, Assignment 11

Mollification

Consider $\mu_1 : \mathbb{R} \to \mathbb{R}$ by

$$\mu_1(x) = \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

It can be shown that $\mu_1 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \mu_1 = [-1, 1]$. A plot of μ_1 is shown in Figure 1.

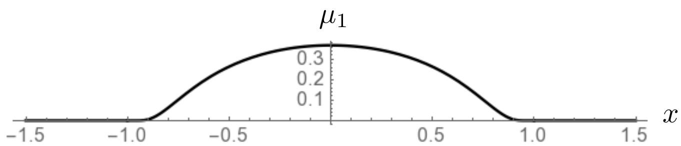


Figure 1: The Test Function μ_1.

1. (a) Use mathematical software to make your own plot of μ_1.
   (b) Use μ_1 to construct a function $\mu_\delta \in C_c^\infty(\mathbb{R})$ with

   $$\operatorname{supp} \mu_\delta = [-\delta, \delta] \qquad \text{and} \qquad \int \mu_\delta = 1.$$

   (c) Consider $\mu_\delta * f : \mathbb{R} \to \mathbb{R}$ for $f \in L^1_{loc}(\mathbb{R})$ by

   $$\mu_\delta * f(x) = \int_{\xi \in \mathbb{R}} \mu_\delta(\xi) f(x - \xi).$$

   This is called the **convolution** (integral) of μ_δ and f. Show

   $$\mu_\delta * f(x) = f * \mu_\delta(x) = \int_{\xi \in \mathbb{R}} f(\xi) \mu_\delta(x - \xi), \tag{1}$$

   that is, convolution is commutative.
   (d) Use (1) to show $\mu_\delta * f \in C^\infty(\mathbb{R})$.
   (e) Determine the support of $h = \mu_\delta * \chi_{[0,\infty)}$ and the support of h'. The particular characteristic function $H = \chi_{[0,\infty)}$ is called the **Heaviside** (step) **function**.
   (f) Plot h using various values of $\delta > 0$ using mathematical software. If you're using Mathematica, the function `Piecewise` can be very useful for this. There's even a function `Convolve` that can be used, but I wasn't able to get it to work properly and had better luck simply using `NIntegrate` over the interval $(-\delta, \delta)$. One other Mathematica function (name) which might be useful to know/look up is `HeavisideTheta`.
   (g) Plot h by hand for general $\delta > 0$.

Weak Derivatives

2. Consider (once again) $u \in W^{1,1}(a,b)$. This means $u \in L^1(a,b)$ and u has a weak derivative $g \in L^1(a,b)$.

   Take two points x_1 and x_2 in (a,b). Assume $x_1 < x_2$. Let μ_δ be a non-negative even C_c^∞ function with support $[-\delta,\delta]$ and $\int \mu_\delta = 1$ discussed in the previous problem. Consider the function $\phi \in C_c^\infty(\mathbb{R})$ obtained by integrating

   $$\phi'(x) = \mu_\delta(x - x_1) - \mu_\delta(x - x_2).$$

   (a) Let $x_1 = -1/2$ and $x_2 = 1/2$. Use numerical software to plot the graph of ϕ and ϕ'.

   (b) Show

   $$[\mu_\delta * (\chi_{[x_1,\infty)} - \chi_{[x_2,\infty)})](x) = \int_{-\infty}^{x} [\mu_\delta(\xi - x_1) - \mu_\delta(\xi - x_2)]\, d\xi.$$

   (c) Plot ϕ and ϕ' by hand (for any $x_1 < x_2$).

   (d) Using ϕ as a test function show

   $$-\int_{x\in\mathbb{R}} u(x)[\mu_\delta(x - x_1) - \mu_\delta(x - x_2)] = \int g\mu_\delta * (\chi_{[x_1,\infty)} - \chi_{[x_2,\infty)}).$$

   (e) Recall the following definition: A point $x_0 \in (a,b)$ is a **Lebesgue point** of $u \in L^1(a,b)$ if

   $$\lim_{\delta\searrow 0} \frac{1}{2\delta} \int_{x\in(x_0-\delta,x_0+\delta)} |u(x) - u(x_0)| = 0.$$

   Given that x_1 and x_2 are Lebesgue points of u and ϕ as described above, show

   $$\lim_{\delta\to 0} \int u\phi' = u(x_1) - u(x_2) \qquad \text{and} \qquad \lim_{\delta\to 0} \int g\phi = \int_{(x_1,x_2)} g.$$

   (f) Recall the **Lebesgue lemma**: Given any $f \in L^1(a,b)$ and any $\epsilon > 0$, there is some $\delta > 0$ such that

   $$\left.\begin{array}{c} A \subset (a,b) \text{ measurable} \\ \mu A < \delta \end{array}\right\} \qquad \Longrightarrow \qquad \int_A |f| < \epsilon.$$

   Use this result to show $u \in C^0(\mathcal{A})$ where

   $$\mathcal{A} = \{x \in (a,b) : x \text{ is a Lebesgue point for } u\}.$$

   (g) **Conjecture** If $u \in L^1(a,b)$ is continuous on a set $A \subset (a,b)$ with $\mu A = b - a$, then there exists a continuous extension $\overline{u} \in C^0(a,b)$ with

   $$\overline{u}\big|_{x\in A} \equiv u.$$

   Were this conjecture correct, then since we know almost every point is a Lebesgue point for an L^1 function, the preceeding parts of this problem would consitute a proof that $W^{1,1}(a,b) \subset C^0(a,b)$ which, of course, is true. The conjecture, however, is false. Give a counterexample.

Fundamental Solution

Consider (once again) the two point boundary value problem for the 1D Poisson equation:

$$\begin{cases} -\Delta u = -u'' = f \\ u(a) = a(b) \end{cases} \tag{2}$$

where this time $f \in C^2[a,b]$.

3. (a) Show there exists an extension $\overline{f} \in C_c^2(\mathbb{R})$ of f. Hint: Because $f \in C^2[a,b]$, there is some $\delta > 0$ and an extension $f_0 \in C^2(a-\delta, b+\delta)$. Mollify $\chi_{(a-\delta/2,b+\delta/2)}$ to obtain a function $\phi \in C_c^\infty(\mathbb{R})$ with $\phi(x) \equiv 1$ for $x \in [a,b]$. Multiply f_0 by ϕ when both are defined. (There could, of course, be other extensions.)

   (b) Let $u_0(x) = \Phi * \overline{f}$ where $\Phi = -|x|/2$ is the fundamental solution for the operator $-\Delta u = -u''$. Show $u_0 \in C^2(\mathbb{R})$ and calculate $-\Delta u_0 = -u_0''$. Hint:

$$-u_0''(x) = -\Phi * \overline{f}''.$$

   (c) Use u_0 to solve (2) in the form $u(x) = u_0(x) - w(x)$ where $w''(0) = 0$, $w(a) = u_0(a)$, and $w(b) = u_0(b)$.

Solution:

(a) Take f_0 as in the hint, and let $\phi = \mu_{\delta/4} * \chi_{(a-\delta/2,b+\delta/2)}$.

$$\bar{f}(x) = \begin{cases} \phi(x) f_0(x), & x \in (a - 3\delta/4, b + 3\delta/4) \\ 0, & x \notin (a - 3\delta/4, b + 3\delta/4). \end{cases}$$

(b)

$$\begin{aligned}
-u_0''(x) &= -\Phi * \overline{f}''(x) \\
&= -\int_{\xi\in\mathbb{R}} \Phi(\xi)\, \overline{f}''(x-\xi) \\
&= \frac{1}{2}\int_{\xi\in\mathbb{R}} |\xi|\, \overline{f}''(x-\xi) \\
&= -\frac{1}{2}\int_{-\infty}^{0} \xi\, \overline{f}''(x-\xi)\, d\xi + \frac{1}{2}\int_0^\infty \xi\, \overline{f}''(x-\xi)\, d\xi \\
&= -\frac{1}{2}\left[-\xi \overline{f}'(x-\xi)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} \overline{f}'(x-\xi)\, d\xi \right] + \frac{1}{2}\left[-\xi\overline{f}'(x-\xi)\Big|_0^\infty + \int_0^\infty \overline{f}'(x-\xi)\, d\xi \right] \\
&= -\frac{1}{2}\left[-\overline{f}(x-\xi)\Big|_{-\infty}^{0} \right] + \frac{1}{2}\left[-\overline{f}(x-\xi)\Big|_0^\infty \right] \\
&= \frac{1}{2}\overline{f}(x) + \frac{1}{2}\overline{f}(x) \\
&= \overline{f}(x).
\end{aligned}$$

(c)

$$w(x) = \frac{u_0(b) - u_0(a)}{b-a}(x-a) + u_0(a) = \frac{u_0(b) - u_0(a)}{b-a}\,x + \frac{bu_0(a) - au_0(b)}{b-a}.$$

A Path Integral

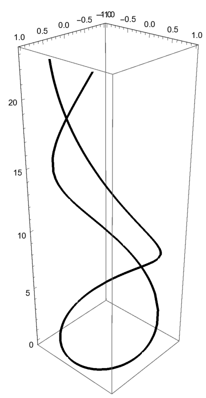
4. Consider the curve

$$\Gamma = \{(\cos t, \sin t, t^2/2) : t \in [0, 2\pi]\}.$$

Compute

$$\int_\Gamma f$$

where $f = f(x, y, z) = x^2 + y^2 + 2z$. **This** may not be as easy as it looks at first. Incidentally, this is sort of an interesting curve which is a variation on the theme of a helix. Here is a plot for $-2\pi - 0.5 < t < 2\pi + 0.5$:



Solution:

(a) $\gamma(t) = (\cos t, \sin t, t^2/2)$.

(b) $\gamma'(t) = (-\sin t, \cos t, t)$, so $\sigma = \sqrt{1+t^2} = |\gamma'(t)|$.

(c) First of all

$$\int_\Gamma f = \int_0^{2\pi} (1+t^2)\sqrt{1+t^2}\,dt = \int_0^{2\pi} \sqrt{1+t^2}\,dt + \int_0^{2\pi} t^2\sqrt{1+t^2}\,dt.$$

The first integral should have something to do with an inverse sinh, and the second one should succumb to integration by parts:

$$\begin{aligned}\int_0^{2\pi} \sqrt{1+t^2}\,dt &= \int_0^{2\pi} \frac{1}{\sqrt{1+t^2}}\,dt + \int_0^{2\pi} \frac{t^2}{\sqrt{1+t^2}}\,dt \\ &= \sinh^{-1}(2\pi) + t\sqrt{1+t^2}\Big|_{t=0}^{2\pi} - \int_0^{2\pi} \sqrt{1+t^2}\,dt \\ &= \sinh^{-1}(2\pi) + 2\pi\sqrt{1+4\pi^2} - \int_0^{2\pi} \sqrt{1+t^2}\,dt.\end{aligned}$$

Consequently,

$$\int_0^{2\pi} \sqrt{1+t^2}\,dt = \frac{1}{2}\sinh^{-1}(2\pi) + \pi\sqrt{1+4\pi^2}.$$

For the other one

$$\begin{aligned}\int_0^{2\pi} t^2\sqrt{1+t^2}\,dt &= \frac{t}{3}(1+t^2)^{3/2}\Big|_{t=0}^{2\pi} - \frac{1}{3}\int_0^{2\pi} (1+t^2)^{3/2}\,dt \\ &= \frac{2\pi}{3}(1+4\pi^2)^{3/2} - \frac{1}{3}\int_\Gamma f.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Gamma f &= \frac{3}{4}\left[\frac{1}{2}\sinh^{-1}(2\pi) + \pi\sqrt{1+4\pi^2} + \frac{2\pi}{3}(1+4\pi^2)^{3/2}\right] \\ &= \frac{1}{8}\left[3\sinh^{-1}(2\pi) + 2\pi\left(3\sqrt{1+4\pi^2} + 2(1+4\pi^2)^{3/2}\right)\right] \\ &= \frac{1}{8}\left[3\sinh^{-1}(2\pi) + 2\pi\left(5+8\pi^2\right)\sqrt{1+4\pi^2}\right].\end{aligned}$$

Gradient Field §6.7-11 (Boas)

In Assignment 5 Problems 1-4 we considered the gradient PDEs in the plane.

5. A vector field $\mathbf{v} = (\phi, \psi)$ on $\mathbb{R}^2$ is a **gradient field** or **exact** or **conservative** if there exists a **potential function** $u : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{v} = Du$.
   (a) Extend $\mathbf{v}$ to a field $\overline{\mathbf{v}} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{v}(x, y, z) = (v_1, v_2, 0)$. Interpret the condition for $\mathbf{v}$ to be a gradient field from Assignment 5 Problem 4 in terms of the **curl operator** applied to $\overline{\mathbf{v}}$.
   (b) What is a natural domain and codomain for the curl operator?
   (c) Give a counterexample to the following assertion: If $\mathbf{v} \in C^1(U)$ and $\operatorname{curl} \mathbf{v} \equiv \mathbf{0}$ on U then there exists a function $u \in C^2(U)$ such that $Du = \mathbf{v}$.

Vector Valued Functions

6. (6.4.6) If a charged particle moves in the plane with path given by $\mathbf{r} : \mathbb{R} \to \{(x(t), y(t), 0) : t \in \mathbb{R}\}$ according to Newton's second law with $\mathbf{F} = q[\mathbf{v} \times (0, 0, b)]$ with b constant, show $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{F}$ are perpendicular and both have constant magnitude.