Assignment 11:
Integration and Laplace's PDE
Due Wednesday, April 12, 2023

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Problem 1 (More solutions of Laplace's PDE) Let $L, M, j > 0$ with $j \in \mathbb{N}$. Consider the functions

$$A_j(x) = \sin\left(\frac{j\pi x}{L}\right) \qquad \text{and} \qquad B_j(y) = \sinh\left(\frac{j\pi y}{L}\right) \Big/ \sinh\left(\frac{j\pi M}{L}\right).$$

(a) Plot $A_j : [0, L] \to \mathbb{R}$ and $B_j : [0, M] \to \mathbb{R}$. Use mathematical software if necessary.

(i) What Sturm-Liouville problem does A satisfy?

(ii) What Sturm-Liouville problem does B satisfy?

(b) Use mathematical software to plot $u_j : [0, L] \times [0, M] \to \mathbb{R}$ by $u_j(x, y) = A_j(x)B_j(y)$.

(c) Show $u_j : [0, L] \times [0, M] \to \mathbb{R}$ by $u_j(x, y) = A_j(x)B_j(y)$ is a solution of Laplace's equation on the rectangle and write down the boundary values satsified by u.

Problem 2 (Fourier's solution) Fourier considered the implications of Problem 1 above:

The function $v_k : [0, L] \times [0, M] \to \mathbb{R}$ by

$$v_k(x, y) = \sum_{j=1}^{k} a_j \sin\left(\frac{j\pi x}{L}\right) \sinh\left(\frac{j\pi y}{L}\right)$$

is a solution of Laplace's equation with boundary values

$$(v_k)\Big|_{\partial U\setminus\Gamma} \equiv 0$$

where $U = (0, L) \times (0, M)$ and $\Gamma = \{(x, M) : x \in [0, L]\}$ and

$$v_k(x, M) = \sum_{j=1}^{k} \left[a_j \sinh\left(\frac{j\pi M}{L}\right)\right] \sin\left(\frac{j\pi x}{L}\right).$$

This means Fourier (and you) could solve any boundary value problem for Laplace's equation on a rectangle with boundary values along the sides given as linear combinations of sine solutions of the Sturm-Liouville problem

$$\begin{cases} A'' = -\lambda A, & x \in (0, L) \\ A(0) = A(L) = 0. \end{cases}$$

This was Problem 10 of Assignment 10. And this means you can solve "a lot" of boundary value problems for the Laplace equation on a rectangle... but not all of them.

Consider (along with Fourier) the general boundary value problem for Laplaces' equation on a rectangle:

$$\begin{cases} \Delta u = 0, & (x, y) \in U \\ u\Big|_{\partial U} \equiv f \end{cases} \tag{1}$$

where $f \in C^0(\partial U)$.

(a) Assume (you can solve the four boundary value problems)

$$\begin{cases} \Delta w_\ell = 0, & (x, y) \in U \\ (w_\ell)\Big|_{\partial U\setminus\Gamma_\ell} \equiv 0 \\ \\ (w_\ell)\Big|_{\Gamma_\ell} \equiv f\Big|_{\Gamma_\ell} \end{cases} \tag{2}$$

where

$$\Gamma_1 = \{(x, 0) : 0 \le x \le L\},$$
$$\Gamma_2 = \{(L, y) : 0 \le y \le M\},$$
$$\Gamma_3 = \{(x, M) : 0 \le x \le L\},$$
$$\Gamma_4 = \{(0, y) : 0 \le y \le M\},$$

and $f \in C^0(\partial U)$, and the solutions are

$$w_\ell \in C^2(U) \cap C^0(\overline{U}) \qquad \text{for} \qquad \ell = 1, 2, 3, 4.$$

What is the solution of the general boundary problem (1) for Laplace's equation?

(b) Under what condition(s) do you expect the hypothesis/assumption of part **(b)** is likely to be reasonable/true?

(c) Let's focus on problem (2) with $\ell = 3$. You should believe by now that the other problems for $\ell \neq 3$ should be similar. Assume there is a solution given by a (convergent) series

$$w(x, y) = \sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi x}{L}\right) \sinh\left(\frac{j\pi y}{L}\right). \tag{3}$$

with

$$w(x, M) = \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi x}{L}\right). \tag{4}$$

(i) Let $g : [0, L] \to \mathbb{R}$ by

$$g(x) = f(x, M).$$

Write down the boundary condition on Γ_3 of (2) when $\ell = 3$ in terms of the series in (4) and the function g.

(ii) Multiply both sides of the relation you got in part **(i)** by the basis function

$$\sin\left(\frac{m\pi x}{L}\right)$$

and integrate the result on $[0, L]$.

(iii) Conclude that we must have

$$b_m = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx \qquad \text{for} \qquad m = 1, 2, 3, \ldots.$$

These numbers are called the **Fourier** (sine) **coefficients** of the function g.

The function w from (3) resulting from choosing the coefficients b_j, $j = 1, 2, 3, \ldots$ in part **(iii)** above is called **Fourier's solution** of the problem (2) for $\ell = 3$.

Problem 3 (Problem 2 above)

(a) Find Fourier's solution for the problem

$$\begin{cases} \Delta w = 0, & (x,y) \in U = (0,2) \times (0,3) \\ w(x,0) = w(0,y) = w(L,y) \equiv 0, & (x,y) \in [0,2] \times [0,3] \\ w(x,M) = 1 - (x-1)^2. \end{cases}$$

(b) Plot the first k terms your solution gives for the value of $w(x,M)$ along with/compared to the actual values of the function $1-(x-1)^2$ for $k = 1,2,3,4$.

(c) Plot the first k terms of your solution for $k = 1,2,3,4$.

Problem 4 (Problem 3 above)

(a) Find Fourier's solution for the problem

$$\begin{cases} \Delta w = 0, & (x,y) \in U = (0,2) \times (0,3) \\ w(x,0) = w(0,y) = w(L,y) \equiv 0, & (x,y) \in [0,2] \times [0,3] \\ w(x,M) = (x-1)^2. \end{cases}$$

(b) Plot the first k terms of the value you get for $w(x,M)$ along with the actual boundary value function $(x-1)^2$ for $k = 1,2,3,4$.

(c) Plot the first k terms of your solution for $k = 1,2,3,4$.

Problem 5 (Problem 4 above)

(a) Find Fourier's solution for the problem

$$\begin{cases} \Delta w = 0, & (x,y) \in U = (0,2) \times (0,3) \\ w(x,0) = w(0,y) = w(L,y) \equiv 0, & (x,y) \in [0,2] \times [0,3] \\ w(x,M) = h(x-1) \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$ is the Heaviside function.

(b) Plot the first k terms of the value you get for $w(x,M)$ along with $h(x-1)$ for $k = 1,2,3,4$.

(c) Plot the first k terms of your solution for $k = 1,2,3,4$.

Problem 6 (Problems 1-5 above) Make some guesses about the regularity of solutions of Laplace's PDE and the relation of this regularity and the regularity of the boundary values.

Problem 7 (weak solutions of Laplace's equation) Given an open set $U \subset \mathbb{R}^2$ a function $u \in C^0(U)$ is said to be a **continuous interior weak solution** of Laplace's equation if

$$\int_U u \Delta \phi = 0$$

for every $\phi \in C_c^\infty(U)$. Show that every classical solution of Laplace's equation $\Delta u = 0$ is a weak solution.

Problem 8 (Poisson's equation) Given an open set $U \subset \mathbb{R}^n$ and a function $f \in C^0(U)$, Poisson's equation is

$$\Delta u = f.$$

Assume $F : \mathbb{R}^n \to \mathbb{R}$ satisfies $F \in C^2(\mathbb{R}^n)$. Assume further that you can solve the boundary value problem

$$\begin{cases} \Delta w = F, & \mathbf{x} \in U \\ w\big|_{\partial U} \equiv 0 \end{cases}$$

for a solution $w \in C^2(\overline{U})$.

Formulate a boundary value problem for Laplace's equation on U (with inhomogeneous boundary values) which you can solve using w.

Problem 9 (weak H^1 solutions of Poisson's equation) Given a bounded open set $U \subset \mathbb{R}^n$, we denote by $H^1(U)$ the collection of all measurable functions satisfying

(i) u is **square integrable** on U, that is

$$\int_U |u|^2 < \infty.$$

Equivalently, we can say $u \in L^2(U)$. The space $L^2(U)$ is precisely the collection of all square integrable measurable functions.

(ii) u has first order weak derivatives $w_j \in L^2(U)$ for $j = 1, 2, \dots, n$. Remember that this means

$$-\int w_j \phi = \int u \frac{\partial \phi}{\partial x_j} \qquad \text{for every } \phi \in C_c^\infty(U).$$

Given $f \in C^0(U)$, we say $u \in H^1(U)$ is a weak solution of the boundary value problem

$$\begin{cases} \Delta u = f, & onU \\ u\big|_{\partial U} = 0 \end{cases} \tag{5}$$

if the following conditions hold

(i)

$$-\int \sum_{j=1}^{n} w_j \frac{\partial \phi}{\partial x_j} = \int_U f\,\phi \qquad \text{for all } \phi \in C_c^\infty(U)$$

where $w_1, w_2, \ldots, w_n$ are the weak first partial derivatives of u.

(ii) There exists a sequence $\{\phi_k\}_{k=1}^\infty \subset C_c^\infty(U)$ with

$$\lim_{k\to\infty} \phi_k = u$$

in the sense that

$$\lim_{k\to\infty} \left(\int_U |\phi_k - u|^2 + \sum_{j=1}^{n} \int_U \left| \frac{\partial \phi_k}{\partial x_j} - w_j \right|^2 \right) = 0.$$

This second condition is the weak formulation of the boundary condition.

Show a classical solution $u \in C^2(\overline{U})$ of the boundary value problem (5) for Poisson's equation is a weak solution.

Problem 10 (mean value property) Let $u \in C^2(U)$ be a classical solution of Laplace's equation in an open set $U \subset \mathbb{R}^2$. Establish the **mean value property** for u as follows:

(a) If $\overline{B_r(\mathbf{p})} \subset U$, compute

$$\frac{d}{dt}\left[\frac{1}{2\pi t}\int_{\partial B_t(\mathbf{p})} u\right]$$

for $0 < t \le r$ and express your answer as an integral over $B_t(\mathbf{p})$. Hint(s): Change variables in the integral so the radius t does not appear in the limit of integration. Differentiate under the integral sign. Use the divergence theorem.

(b) Conclude

$$\frac{d}{dt}\left[\frac{1}{2\pi t}\int_{\partial B_t(\mathbf{p})} u\right] = 0.$$

(c) Conclude

$$u(\mathbf{p}) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{p})} u.$$

This is the mean value property of harmonic functions.