Math 6702, Assignment 10

Weak Derivatives

1. Give an example of $u \in W^1(B_1(0))\backslash C^0(B_1(0))$ where $B_1(0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$ and $n > 1$. Hint: Consider $u(x, y) = u_0(x^2 + y^2) = (x^2 + y^2)^{-\alpha}$ where α is small enough.

Fundamental Solution

Recall Problem 3 of Assignment 9 in which $\Phi = \Phi(x, y)$ satisfied $\Phi(x, y) = \Phi_0(x^2 + y^2)$ and "$-\Delta\Phi = \delta_0$" on $\mathbb{R}^2$ in the distributional sense where δ_0 represents a point impulse at the origin. Again let $U \subset \mathbb{R}^2$ be a bounded open domain with a smooth boundary curve and with $\xi \in U$ fixed, consider

$$v(\mathbf{x}, \xi) = \Phi(\mathbf{x} - \xi).$$

Consider now also the boundary value problem for Laplace's equation:

$$\begin{cases} \Delta w = 0 \text{ on } U \\ w\Big|_{\mathbf{x}\in\partial U} = \Phi(\mathbf{x} - \xi) = v(\mathbf{x}, \xi). \end{cases}$$

Let us assume there exists a unique solution $w = w(\mathbf{x}) = w(\mathbf{x}, \xi)$ of this problem for each fixed $\xi \in U$ with

$$w \in C^\infty(U \times U) \cap C^0(\overline{U} \times U).$$

You may not be able to write down a formula for w, but we have an existence and uniqueness theorem for weak solutions (transitioning the boundary values to Poisson's equation with homogeneous boundary values), and in fact this regularity does hold for those weak solutions, at least if ∂U is smooth. In any case, let's assume it for now. Finally, set

$$G(\mathbf{x}, \xi) = \Phi(\mathbf{x} - \xi) - w(\mathbf{x}) = v(\mathbf{x}, \xi) - w(\mathbf{x}, \xi)$$

and

$$u(\mathbf{x}) = \int_{\xi\in U} G(\mathbf{x}, \xi) f(\xi).$$

2. (a) Compute the distributional Laplacian of u. Hint(s): Let $\phi \in C_c^\infty(U)$. Write out

   $$\int_{x\in U} u(\mathbf{x})\Delta\phi(\mathbf{x}),$$

   and use Fubini's theorem and problem 3 of Assignment 9.

   (b) What boundary value problem is satisfied by u, assuming $u \in C^2(U) \cap C^0(\overline{U})$?

Solution:

$$\begin{aligned}
-\int_{x\in U} u(\mathbf{x})\Delta\phi(\mathbf{x}) &= -\int_{x\in U}\int_{\xi\in U} G(\mathbf{x},\xi)f(\xi)\Delta\phi(\mathbf{x}) \\
&= -\int_{\xi\in U} f(\xi)\int_{\mathbf{x}\in U}[v(\mathbf{x},\xi)-w(\mathbf{x},\xi)]\Delta\phi(\mathbf{x}) \\
&= \int_{\xi\in U} f(\xi)\left\{-\int_{\mathbf{x}\in U} v(\mathbf{x},\xi)\Delta\phi(\mathbf{x}) + \int_{\mathbf{x}\in U} w(\mathbf{x},\xi)\Delta\phi(\mathbf{x})\right\} \\
&= \int_{\xi\in U} f(\xi)\phi(\xi).
\end{aligned}$$

Note: The first integral reduced to $\phi(\xi)$ because $v(\mathbf{x},\xi) = \Phi(\mathbf{x}-\xi)$ is a fundamental solution shifted to the singular unit source at ξ as in the previous problem, and the second integral vanished because w is a classical solution.

This means

$$\begin{cases} -\Delta u = f \text{ on } U \\ u\big|_{\partial U} = 0. \end{cases}$$

Thus, this is a formula for solutions of Poisson's equation with homogeneous boundary conditions in terms of a solution for Laplace's equation with a Dirichlet condition. Naturally, G is called a **Green's function** for the problem.

Surface Integrals

3. Let $\mathbf{v}$ be the vector field $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ on $\mathbb{R}^3$.

   (a) Compute the flux integral

   $$\int_{\partial C} \mathbf{v}\cdot N$$

   where $C = C_r(\mathbf{0}) = \{\mathbf{x}\in\mathbb{R}^3 : \|\mathbf{x}\|_{\ell^1} < r\}$ is the unit ball, i.e., cube, in the ℓ^∞ norm

   $$\|\mathbf{x}\|_{\ell^\infty} = \max|x_j|$$

   and N is the outward unit normal to ∂C.

   (b) Your answer $I = I(r)$ should depend on r. Compute the limit

   $$\lim_{r\searrow 0}\frac{I(r)}{\text{vol}(C_r(\mathbf{0}))}.$$

Vector Valued Functions

4. (6.4.4) If $\mathbf{r} : (a,b)\to\mathbb{R}^3$ with $|\mathbf{r}(t)|\equiv 1$, show that $\mathbf{v} = \mathbf{r}'$ is perpendicular to $\mathbf{r}$.

5. (6.4.10) If $\mathbf{r} : (a, b) \to \mathbb{R}^3$, evaluate

$$\int (\mathbf{r} \times \mathbf{r}'')\, dt.$$

Gradient Field

6. (6.6.11) Assume the temperature in the plane is given by $u(x, y) = x^2 - y^2$. The level curves of u in this case are called **isothermal curves**. Draw the isothermal curves and determine the path along which heat would flow through $(x_0, y_0) = (1, 2)$. Hint: According to Fourier's law, heat flows in the direction $\vec{\phi} = -KDu$, that is, in the opposite direction of the temperature gradient. Does the heat conductivity K (assumed to be constant) effect the path?