Assignment 10:
Integration
Due Wednesday, April 5, 2023

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Problem 1 (beyond Hooke's constant) This is the fourth in a series of problems on Hooke's constant.

If you were able to formulate an alternative model for homogeneous deformations of a spring in Part **(d)** of Problem 1 of Assignment 9, assume the spring is massless but exerts tension/compression force uniformly along the homogeneous deformation with the horizontal model and has free end attached to a (point) mass m that moves without friction under the influence of this force, and

(a) Model the motion of the mass in terms of the horizontal extension u.

(b) Model the motion of the mass attached to one of the halves of the spring if you cut it in half.

Problem 2 (beyond Hooke's constant) Consider the massless spring of the previous problem with a mass attached to the end, but hanging in a downward gravity field and using the vertical model measurement function $v = -u$. Model the motion of the mass.

Problem 3 (Boas Problem 5.2.33) An **areal mass density** is a non-negative real valued function $\rho : A \to [0, \infty)$ on a domain A naturally admitting area measure. The (model) mass of a set A defined to be

$$M = \int_A \rho.$$

Notice the physical dimensions of an areal mass density are given by

$$[\rho] = \frac{M}{L^2}.$$

If A is the quarter disk

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4,\ x > 0,\ y > 0\}$$

in the first quadrant and the mass density it taken to be $\rho(x, y) = x+y$, then calculate the (model) mass.

Problem 4 (Boas Problem 5.2.36) Light impinging on a square mirror partially reflects off the surface of the mirror with some light passing through the glass of the mirror. Assume the mirror surface is modeled by the region

$$C_1(0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_j| \leq 1,\ j = 1, 2\} \subset \mathbb{R}^2.$$

Assume the light incident on the mirror is modeled by an **intensity density** $I : C_1(0, 0) \to [0, \infty)$ according to which

$$\int_{C_1(0,0)} I$$

models the total energy of the light impinging on the mirror, and the reflection of the light is modeled by a **fraction of reflection** density $\alpha : C_1(0, 0) \to \mathbb{R}$ by

$$\alpha(x_1, x_2) = \frac{(x_2 - x_1)^2}{4}$$

so that

$$R = \int_{C_1(0,0)} I\alpha$$

models the total energy of light reflected. If I is constant, find the value of R.

Problem 5 (Boas Problem 5.2.40) Consider the iterated integrals

$$\int_1^2 \int_x^{2x} \int_0^{1/z} z\, dy\, dz\, dx.$$

(i) Find the volume V so that these iterated integrals have value

$$\int_V z.$$

(ii) Express the value as five different iterated integrals.

(iii) Compute the value.

Problem 6 (Problem 5.2.49 in Boas) Calculate $\int_V 1$ where

$$V = \{(x, y, z) : 0 < z < 1 - x^2 - y^2,\ 0 < x, y < x + y < 1\}.$$

Problem 7 (Boas Problems 5.3.17-30) A **linear mass density** $\rho : \Gamma \to [0, \infty)$ is a non-negative function defined on a set/curve Γ subject to length measure, so that the (model) mass of Γ is given by

$$M = \int_\Gamma \rho.$$

If $\Gamma \subset \mathbb{R}^n$, the **center of mass** is defined to be the point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ where

$$\overline{x}_j = \frac{1}{M} \int_\Gamma x_j \rho.$$

Consider

$$\Gamma = \{(x, \sqrt{x}) : 0 \le x \le 2\}.$$

(a) Calculate the mass using the parameterization $\alpha(x) = (x, \sqrt{x})$ when ρ is constant.

(b) Find the length L of Γ.

(c) Parameterize Γ by arclength.

(d) Calculate the mass using the arclength parameterization $\gamma : [0, L] \to \Gamma$ when $\rho(x, \sqrt{x}) = \sqrt{x}$.

(e) Find the **center of mass** when ρ is constant.

Problem 8 (The rationals have zero measure.) Here are some (well-known) sets of numbers:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \qquad \text{(the natural numbers)}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\} \qquad \text{(the integers)}$$

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\} \qquad \text{(the rational numbers)}$$

We have also been using the real numbers $\mathbb{R}$, and it may now be pointed out that $\mathbb{R} \backslash \mathbb{Q}$ is the set of **irrational numbers**.

Theorem 1 (the rationals are countable) There exists a sequence

$$\{q_j\}_{j=1}^{\infty} = \{q_1, q_2, q_3, \ldots\}$$

with

$$\{q_1, q_2, q_3, \ldots\} = \mathbb{Q}.$$

The set/sequence $\{q_1, q_2, q_3, \ldots\}$ is called an **enumeration of the rationals**.

We will not prove Theorem 1, but you can take it as (a) given.

(a) Show that for any $\epsilon > 0$, there exist intervals $I_j = (q_j - r_j, q_j + r_j)$ for $j = 1, 2, 3, \ldots$ such that

$$\sum_{j=1}^{\infty} \mu(I_j) = \sum_{j=1}^{\infty} (2r_j) < \epsilon.$$

(b) Conclude $\mu(\mathbb{Q})$, the measure of the rationals, is zero. Hint(s): If A and B are measurable sets and $A \subset B$, then $\mu(A) \leq \mu(B)$, and if

$$A = \bigcup_{j=1}^{\infty} A_j$$

for some sequence of measurable sets $A_1, A_2, A_3, \ldots$ then

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

Problem 9 (sets of measure zero) Any subset A of $\mathbb{R}$ is said to have **measure zero** if for any $\epsilon > 0$, there exists a countable collection of intervals $I_1, I_2, I_3, \ldots$ such that

$$A \subset \bigcup_{j=1}^{\infty} I_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mu(I_j) < \epsilon,$$

where the measure of an interval is its length.

(a) Show that any countable union

$$A = \bigcup_{j=1}^{\infty} A_j$$

of sets A_j each of which has measure zero also has measure zero: $\mu(A) = 0$.

(b) Is it true that an arbitrary union of sets of measure zero always has measure zero?

Problem 10 (More solutions of Laplace's PDE) In Problems 8-10 of Assignment 8, you found a solution of Laplace's equation $\Delta u = 0$ on a rectangle $U = (0, L) \times (0, M) \subset \mathbb{R}^2$ with **homogeneous boundary conditions**, i.e., $u \equiv 0$ on three sides of the rectangle. Consider the same PDE, Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the same rectangular domain $U = (0, L) \times (0, M) \subset \mathbb{R}^2$.

Given $x_0 > 0$, let us say a function $g : [0, x_0] \to \mathbb{R}$ is a **compatible linear combination** if g is given by

$$g(x) = \sum_{j=1}^{k} a_j \sin\left(\frac{n_j \pi x}{x_0}\right)$$

for some (finitely many) coefficients $a_1, a_2, \ldots, a_k$ and some (finitely many) natural numbers $n_1, n_2, \ldots, n_k$.

Let g_0 and g_1 be compatible linear combinations on the interval $[0, L]$, and let h_0 and h_1 be compatible linear combinations on the interval $[0, M]$. Find a solution $u \in C^2([0, L] \times [0, M])$ of the boundary value problem

$$\begin{cases} \Delta u = 0, & (x, y) \in (0, L) \times (0, M) \\ u(x, 0) = g_0(x), & x \in [0, L] \\ u(x, M) = g_1(x), & x \in [0, L] \\ u(0, y) = h_0(y), & y \in [0, M] \\ u(L, y) = h_1(y), & y \in [0, M] \end{cases}$$

for Laplace's equation. Hint(s): The partial differential equation is **linear** meaning that linear combinations of solutions are solutions:

$$\text{If } \Delta u = \Delta v = 0, \text{ then } \Delta(au + bv) = 0.$$

Consider boundary problems like

$$\begin{cases} \Delta w = 0, & (x,y) \in (0,L) \times (0,M) \\ w(x,0) = 0, & x \in [0,L] \\ w(x,M) = 0, & x \in [0,L] \\ w(0,y) = w_0(y), & y \in [0,M] \\ w(L,y) = 0, & y \in [0,M] \end{cases}$$

with three homogeneous boundary conditions.