Assignment 1: Functions
Due Friday, January 24, 2025

John McCuan

January 5, 2025

Problem 1 (modeling a hanging slinky—step zero) Determine/identify some (interesting) quantity associated with a hanging slinky which you think can be measured and modeled by, i.e., compared to, a real valued function

$$f : (a, b) \to \mathbb{R}$$

of one variable on an open interval (a, b) or possibly on a closed interval $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$. You should identify the real numbers a and b with $a < b$ determing the **interval of definition** though measurements may be needed to give them actual numerical values, and identify the quantity or measurement to which **the values** of the function f should be compared.

When you get done, you should have an idea of exactly what you want to measure and how.

You may wish to change the name of the function f. For example, if you want to compare the values of f to a linear density, then you may want to call the function ρ, λ, or δ, which are more traditional symbols used to denote a linear density. Hint: Do not let f correspond to/model a linear density but rather some quantity which is easier to measure and from which a linear density may be derived.

Let's call your function a **model measurement function**.

Problem 2 (convexity) Find a convex function $f : (-2, 1) \to \mathbb{R}$ satisfying

$$\lim_{x \searrow -2} f(x) = \lim_{x \nearrow 1} f(x) = +\infty.$$

A function $f : (a, b) \to \mathbb{R}$ is **convex** if the inequality

$$f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2) \tag{1}$$

holds whenever $x_1, x_2 \in (a, b)$ and $0 \le t \le 1$.

Problem 3 Draw a picture of the **graph**

$$\{(x, f(x)) : x \in (a, b)\}$$

of a convex function $f : (a, b) \to \mathbb{R}$ illustrating the condition (1).

Problem 4 Is it possible to find an example of a convex function $f : (a, b) \to \mathbb{R}$ that is discontinuous?

Problem 5 Use your example from Problem 2 above to illustrate the value of the **difference quotient**

$$\frac{f(x+h) - f(x)}{h}$$

with $x = 0$ and increment $h = -1$. (Hint: Start your illustration by drawing the graph of f.)

Problem 6 Show the derivative of the **absolute value function** $g : (-\infty, \infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} -x, & x \leq 0 \\ x, & x \geq 0 \end{cases}$$

is not well-defined at $x = 0$. Hint: Show the limit of the difference quotient does not exist as follows:

(a) Assume by way of contradiction that there exists a limit $L \in \mathbb{R}$ for which

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = L.$$

(b) Conclude there is some $\delta > 0$ for which

$$\left| \frac{|h|}{h} - L \right| < 1 \qquad \text{when} \qquad |h| < \delta.$$

(c) Get a contradiction by finding increments h_1 and h_2 satisfying $|h_j| < \delta$ for $j = 1, 2$ and

$$\left| \frac{g(0+h_2) - g(0)}{h_2} - \frac{g(0+h_1) - g(0)}{h_1} \right| \geq 2.$$

Hint hint: Use the triangle inequality.

Problem 7 Compute the **left derivative** $\delta : (-\infty, \infty) \to \mathbb{R}$ given by

$$\delta(x) = \lim_{h \searrow 0} \frac{h(x+h) - h(x)}{h}$$

of the Heaviside function $h : (-\infty, \infty) \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Note, δ is not a real valued function but rather an **extended real valued function** taking values in the **extended real line**, that is $\delta : (-\infty, \infty) \to \mathbb{R} \cup \{\pm\infty\}$.

Problem 8 Recall (or note) the following two definitions: A function $f : (a, b) \to \mathbb{R}$ is **continuous at a point** $x \in (a, b)$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(\xi) - f(x)| < \epsilon \qquad \text{whenever} \qquad |\xi - x| < \delta.$$

A function $f : (a, b) \to \mathbb{R}$ is said to be **continuous on the interval** (a, b) if f is continuous at every point $x \in (a, b)$. In this case we write $f \in C^0(a, b)$. ($C^0(a, b)$ is the set of all real valued functions which are continuous on the interval (a, b).)

The Heaviside function h is continuous at every point $x \in (-\infty, 0) \cup (0, \infty)$, so we could write $h \in C^0((-\infty, 0) \cup (0, \infty))$, but $h \notin C^0(\mathbb{R})$.

Draw the graph of $\sigma : (-\infty, \infty) \to \mathbb{R}$ by

$$\sigma(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and show $\sigma \in C^0(\mathbb{R})$.

Problem 9 Recall the following three definitions: A function $f : (a, b) \to \mathbb{R}$ is **differentiable at a point** $x \in (a, b)$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists (as a finite real number). A function $f : (a, b) \to \mathbb{R}$ is said to be **differentiable on the interval** (a, b) if f is differentiable at every point $x \in (a, b)$. In this case $f' : (a, b) \to \mathbb{R}$ by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is a well-defined function.

Given a function $f : (a, b) \to \mathbb{R}$ which is differentiable on the interval (a, b), we say f is **continuously differentiable** if $f' \in C^0(a, b)$. In this case, we write $f \in C^1(a, b)$.

Find an example of a function $f : (a, b) \to \mathbb{R}$ which is differentiable on the interval (a, b), but is **not** continuously differentiable.

Problem 10 Recall (or note) the following two definitions: A function $f : (a, b) \to \mathbb{R}$ is **increasing** if

$$f(x_2) > f(x_1) \qquad \text{whenever} \qquad a < x_1 < x_2 < b.$$

A function $f : (a, b) \to \mathbb{R}$ is **decreasing** if

$$f(x_2) < f(x_1) \qquad \text{whenever} \qquad a < x_1 < x_2 < b.$$

A function which is increasing or decreasing is said to be (strictly) **monotone**.

Think about the function $f : (a, b) \to \mathbb{R}$ you suggested in Problem 1 above (proposed to be part of modeling a hanging slinky). We have a number of definitions in this assignment concerning continuity, differentiability, monotonicity, and convexity. If your function f turns out to have values which may be reasonably compared to measurements taken from the hanging slinky, what properties do you expect the model measurement function to have (in terms of continuity, differentiability, monotonicity, and convexity)? Also include a discussion of the expected **boundary values** $f(a)$ and $f(b)$. Tell me anything you think should be true about a reasonable model function, i.e., your model function, which you can assert (or think you can assert) without actually making any measurements.

Note: Please give careful attention to Problems 1 and 10 on this assignment. The topics addressed in these problems will come up again.