Lecture 1: Ordinary Differential Equations and The Calculus of Variations Assignment 1 Problem 5 Solution

Assignment Problems Due Monday February 8, 2021

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Problem 1 *(Boas 8.6.1) Find the general solution of* $y'' - 4y = 10$. *Solve the IVP*

$$\begin{cases} y'' - 4y = 10 \\ y(1) = -3, \ y'(1) = -2. \end{cases}$$

Use mathematical software to find a numerical approximation of the solution of the IVP. (Also plot your solution to see the two match.)

Solution: Setting $y_h = e^{\alpha t}$ we get $y_h'' - 4y_h = (\alpha^2 - 4)e^{\alpha t}$. In this way we see the general solution of the homogeneous equation $y_h'' + 4y_h = 0$ is $y_h = ae^{-2t} + be^{2t}$ or equivalently $y_h = a\cosh(2t) + b\sinh(2t)$. Since a particular solution for the original equation is $y_p = -5/2$, the general solution of the original equation is

$$y = y_h + y_p = a\cosh(2t) + b\sinh(2t) - 5/2.$$

The initial conditions require

$$\begin{cases} a\cosh 2 + b\sinh 2 = -1/2 \\ a\sinh 2 + b\cosh 2 = -2. \end{cases}$$

By Cramer's rule

$$a = 2\sinh 2 - (1/2)\cosh 2 \qquad \text{and} \qquad b = (1/2)\sinh 2 - 2\cosh 2.$$

Therefore,

$$y = [2\sinh 2 - (1/2)\cosh 2]\cosh(2t) + [(1/2)\sinh 2 - 2\cosh 2]\sinh(2t) - 5/2.$$

Problem 2 *(Boas 8.6.28) Consider the following ordinary differential operators on complex valued functions of a real variable:*

$$\frac{d}{dt} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \frac{d}{dt}\, u = u'$$

and

$$\mathrm{id} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \mathrm{id}\, u = u.$$

(a) *Expand the linear constant coefficient operator*

$$L : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad Lu = \left(\frac{d}{dt} - a\,\mathrm{id}\right)\left(\frac{d}{dt} - b\,\mathrm{id}\right) u$$

where a and b are complex numbers to obtain an expression of the form $Lu = u'' + pu' + qu$ for complex numbers p and q.

(b) *Find the general solution of $Lu = ke^{ct}$ where k and c are complex numbers by solving $y' - ay = ke^{ct}$ first and then solving $u' - bu = y$ (as linear first order ODEs) in the three cases:*

- **(i)** *$c \neq a$ and $c \neq b$.*
- **(ii)** *$a \neq b$ and $c = a$.*
- **(iii)** *$a = b = c$.*

Solution:

(a)

$$\left(\frac{d}{dt} - a\,\mathrm{id}\right)\left(\frac{d}{dt} - b\,\mathrm{id}\right) u = \left(\frac{d}{dt} - a\,\mathrm{id}\right)(u' - bu) = u'' - bu' - au' + abu.$$

That is, $Lu = u'' - (a+b)u + abu$. Note that if p and q are any complex numbers, we can take the principal (complex) square root and set

$$a = \frac{-p - \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad b = \frac{-p + \sqrt{p^2 - 4q}}{2}.$$

Then we will have $p = -(a+b)$ and $q = ab$.

(b) $u'' - (a+b)u + abu = ke^{ct}$. It is convenient to use the **linear existence and uniqueness theorem** to assert that all equations considered here have solutions existing for all time.

We consider the associated homogeneous equation $u_h'' - (a+b)u_h' + abu_h = 0$ first. The factorization suggests we find y_h with

$$y_h' - ay_h = 0.$$

That is, $y_h = y_h(0)e^{at}$. We then consider

$$u_h' - bu_h = y_h = y_h(0)e^{at}. \tag{1}$$

Notice that the value $y_h(0)$ may be assumed to exist according to the comment above. From (1) we get

$$\left(u_h e^{-bt}\right)' = y_h(0)e^{(a-b)t}.$$

There are various cases we need to consider at this point. If $b \neq a$, then

$$\begin{aligned}
u_h &= \left[u_h(0) + y_h(0)\int_0^t e^{(a-b)\tau}\,d\tau\right]e^{bt} \\
&= \left[u_h(0) + \frac{y_h(0)}{a-b}\left(e^{(a-b)t} - 1\right)\right]e^{bt} \\
&= u_h(0)\,e^{bt} + \frac{y_h(0)}{a-b}\left(e^{at} - e^{bt}\right) \\
&= \left(u_h(0) - \frac{y_h(0)}{a-b}\right)e^{bt} + \frac{y_h(0)}{a-b}\,e^{at}.
\end{aligned}$$

In view of the fact that $y_h(0)$ and $u_h(0)$ may be chosen arbitrarily, we find that the general homogeneous solution has the form

$$u_h = \alpha e^{at} + \beta e^{bt} \qquad \text{where } \alpha \text{ and } \beta \text{ are arbitrary complex constants.}$$

If $a = b$, then the integration proceeds differently:

$$\begin{aligned}
u_h &= \left[u_h(0) + y_h(0)\int_0^t 1\,d\tau\right]e^{bt} \\
&= \left[u_h(0) + y_h(0)\,t\right]e^{bt}.
\end{aligned}$$

Therefore, when $a = b$

$$u_h = (\beta t + \alpha)e^{at} \qquad \text{where } \alpha \text{ and } \beta \text{ are arbitrary complex constants.}$$

We next attempt to apply the same approach to finding a particular solution u_p of the inhomogeneous equation starting with the preliminary equation

$$y_p' - ay_p = ke^{ct}.$$

As before,

$$\left(y_p e^{-at}\right)' = ke^{(c-a)t}, \qquad \text{and} \qquad y_p = \left[y_p(0) + k\int_0^t e^{(c-a)\tau}\,d\tau\right] e^{at}. \tag{2}$$

Here we encounter the cases outlined by Boas:

(i) $c \neq a$ and $c \neq b$. In this case,

$$y_p = \left[y_p(0) + \frac{k}{c-a}\left(e^{(c-a)t} - 1\right)\right] e^{at} = \left(y_p(0) - \frac{k}{c-a}\right) e^{at} + \frac{k}{c-a}\, e^{ct}.$$

Since we are seeking a particular solution, we may choose $y_p(0) = k/(c-a)$ and

$$y_p = \frac{k}{c-a}\, e^{ct}.$$

With this choice, if $u_p' - bu_p = y_p$ we get

$$\left(u_p e^{-bt}\right)' = \frac{k}{c-a}\, e^{(c-b)t}.$$

Thus,

$$\begin{aligned} u_p &= \left[u_p(0) + \frac{k}{(c-a)(c-b)}\left(e^{(c-b)t} - 1\right)\right] e^{bt} \\ &= \left(u_p(0) - \frac{k}{(c-a)(c-b)}\right) e^{bt} + \frac{k}{(c-a)(c-b)}\, e^{ct}. \end{aligned}$$

Again, we may choose $u_p(0)$ for our convenience and obtain a particular solution of the form

$$u_p = \frac{k}{(c-a)(c-b)}\, e^{ct},$$

so the general solution is $u = u_h + u_p$ given by

$$u = \alpha e^{at} + \beta e^{bt} + \frac{k}{(c-a)(c-b)}\, e^{ct} \qquad \text{if } a \neq b$$

and

$$u = (\beta t + \alpha) e^{at} + \frac{k}{(c-a)(c-b)}\, e^{ct} \qquad \text{if } a = b.$$

(ii) $a \neq b$ and $c = a$. In this case, (2) becomes

$$y_p = \left[y_p(0) + k \int_0^t 1 \, d\tau \right] e^{at} = (y_p(0) + kt) e^{at}.$$

Postponing momentarily a specific choice for $y_p(0) = 0$, we consider next

$$u_p' - b u_p = y_p = y_p(0) e^{at} + k t e^{at}.$$

Noting the assumption $a \neq b$, we get

$$\left(u_p e^{-bt}\right)' = y_p(0) e^{(a-b)t} + k t e^{(a-b)t}.$$

Therefore,

$$\begin{aligned}
u_p &= \left[u_p(0) + y_p(0) \int_0^t e^{(a-b)\tau} \, d\tau + k \int_0^t \tau e^{(a-b)\tau} \, d\tau \right] e^{bt} \\
&= \left[u_p(0) + \frac{y_p(0)}{a-b} \left(e^{(a-b)t} - 1 \right) + \frac{k}{a-b} \left(t e^{(a-b)t} - \int_0^t e^{(a-b)\tau} \, d\tau \right) \right] e^{bt} \\
&= \left(u_p(0) - \frac{y_p(0)}{a-b} \right) e^{bt} + \frac{y_p(0)}{a-b}\, e^{at} + \frac{k}{a-b} t e^{at} - \frac{1}{(a-b)^2} \left(e^{at} - e^{bt} \right) \\
&= \left(u_p(0) - \frac{y_p(0)}{a-b} + \frac{1}{(a-b)^2} \right) e^{bt} + \left(\frac{y_p(0)}{a-b} - \frac{1}{(a-b)^2} \right) e^{at} + \frac{k}{a-b}\, t e^{at}.
\end{aligned}$$

Taking $y_p(0) = 1/(a-b)$ and $u_p(0) = 0$, we obtain a particular solution

$$u_p = \frac{k}{a-b}\, t e^{at}$$

and general solution

$$u = \alpha e^{at} + \beta e^{bt} + \frac{k}{a-b}\, t e^{at} = \left(\alpha + \frac{kt}{a-b} \right) e^{at} + \beta e^{bt}.$$

(iii) $a = b = c$. We begin again with (2) which as in the previous case gives

$$y_p = (y_p(0) + kt)e^{at}.$$

The final integration is much easier. We have

$$u_p' - au_p = y_p = y_p(0)e^{at} + kte^{at},$$

so

$$\left(u_p e^{-at}\right)' = y_p(0) + kt.$$

Therefore, taking $u_p(0) = y_p(0) = 0$

$$u_p = \left[u_p(0) + y_p(0)\, t + \frac{k}{2}\, t^2\right] e^{at} = \frac{k}{2}\, t^2 e^{at},$$

and using the homogeneous solution in the case $a = b$

$$u = (\beta t + \alpha)e^{at} + \frac{k}{2}\, t^2 e^{at} = \left(\frac{k}{2}\, t^2 + \beta t + \alpha\right) e^{at}.$$

Problem 3 *(Boas 8.7.5) The shape of a hanging chain is modeled by solutions of*

$$(y'')^2 = k^2[1 + (y')^2].$$

Find the general solution of this (nonlinear) ODE.

Solution: We can write the equation as

$$\frac{y''}{\sqrt{1 + y'^2}} = k,$$

or

$$\frac{d}{dx} \sinh^{-1} y' = k.$$

Thus, Integrating once yields

$$\sinh^{-1} y' = kx + b$$

or

$$y' = \sinh(kx + b).$$

It follows that $y(x) = \cosh(kx + b) + c$ for some constants b and c.

Problem 4 *(Boas 8.7.6) The* **signed curvature** *of the graph of a function $u \in C^2[a,b]$ at the point $(x, u(x))$ is defined to be the derivative*

$$k = \frac{d\psi}{ds}$$

with respect to arclength

$$s = \int_a^x \sqrt{1 + [u'(\xi)]^2}\, d\xi$$

of the inclination angle ψ defined by

$$(\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1+[u'(x)]^2}}, \frac{u'(x)}{\sqrt{1+[u'(x)]^2}} \right).$$

(a) *Find the curvature of the graph of $u(x) = \sqrt{r^2 - x^2}$ for $|x| < r$.*

(b) *Find the curvature of the graph of $u(x) = -\sqrt{r^2 - x^2}$ for $|x| < r$.*

(c) *Show the curvature is given in general by*

$$k = \frac{u''}{(1 + [u'(x)]^2)^{3/2}}.$$

(d) *Solve the ODE*

$$\frac{u''}{(1 + [u'(x)]^2)^{3/2}} = c$$

where c is a (real) constant.

Solution:

(a) Find the curvature of the graph of $u(x) = \sqrt{r^2 - x^2}$ for $|x| < r$.

Notice that

$$u' = -\frac{x}{\sqrt{r^2 - x^2}}.$$

Also, differentiating the arclength s as a function of x, we have

$$\frac{ds}{dx} = \sqrt{1 + u'^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}} \geq 1 > 0.$$

This means, we can think of this semicircle as parameterized by the arclength, and in particular, we can think of x as a function of the arclength s as well with

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+u'^2}} = \frac{\sqrt{r^2-x^2}}{r} = \sqrt{1-(x/r)^2}.$$

We note also that in general

$$\frac{dx}{ds} = \cos\psi$$

Therefore,

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}} = -\frac{x}{\sqrt{r^2-x^2}}\,\frac{\sqrt{r^2-x^2}}{r} = -\frac{x}{r}.$$

By the chain rule then

$$\frac{d}{ds}\sin\psi = \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right)\frac{dx}{ds} = \frac{u''}{(1+u'^2)^{3/2}}\cos\psi.$$

On the other hand,

$$\frac{d}{ds}\sin\psi = \cos\psi\,\frac{d\psi}{ds} = k\,\cos\psi$$

where k is the curvature. Putting these two expressions together and canceling the cosine, we get

$$k = \frac{u''}{(1+u'^2)^{3/2}} = \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = \frac{d}{dx}\left(-\frac{x}{r}\right) = -\frac{1}{r}.$$

The sign is somewhat crucial here. Another approach is to complete the integration to obtain the arclength as an explicit function of x:

$$s = \int_0^x \frac{1}{\sqrt{1-(\xi/r)^2}}\,d\xi = r\sin^{-1}(x/r)$$

so that

$$x = r\sin(s/r) \qquad \text{and} \qquad \psi = -\frac{s}{r}.$$

Then, as above $k = d\psi/ds = -1/r$. (To get the expression for ψ, write down a parameterization

$$\gamma(s) = (r\sin(s/r), \sqrt{r^2 - r^2\sin^2(s/r)}) = r(\sin(s/r), r\cos(s/r)).$$

Then $(\cos\psi, \sin\psi) = \dot\gamma(s) = (\cos(s/r), -\sin(s/r)) = (\cos(-s/r), \sin(-s/r)).$)

(b) Find the curvature of the graph of $u(x) = -\sqrt{r^2 - x^2}$ for $|x| < r$. The difference is that $u' = x/\sqrt{r^2 - x^2}$ so $u'/\sqrt{1 + u'^2} = x/r$, $\psi = s/r$ and

$$k = \frac{d}{dx}\left(\frac{x}{r}\right) = \frac{d}{ds}\left(\frac{s}{r}\right) = \frac{1}{r}.$$

(c) Show the curvature is given in general by

$$k = \frac{u''}{(1 + [u'(x)]^2)^{3/2}}.$$

I already did this above in part (a).

(d) Solve the ODE

$$\frac{u''}{(1 + [u'(x)]^2)^{3/2}} = c$$

where c is a (real) constant.

We can write this equation as

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1 + u'^2}}\right) \frac{d}{dx} \sin\psi = c.$$

Therefore,

$$\frac{u'}{\sqrt{1 + u'^2}} = cx + d. \tag{3}$$

We note here that we must have $|cx + d| \leq 1$. As long as $c \neq 0$, this condition determines an interval with endpoints $(1 - d)/c$ and $-(1 + d)/c$ of length $2/|c|$. Let us call the endpoints of this interval $-d/c - r$ and $-d/c + r$ where $r = 1/|c|$.

Squaring in (3), we get $u'^2 = (cx + d)^2(1 + u'^2)$ or

$$u' = \frac{cx + d}{\sqrt{1 - (cx + d)^2}}.$$

Therefore,

$$\begin{aligned} u &= u(-d/c) + \int_{-d/c}^{x} \frac{c\xi + d}{\sqrt{1 - (c\xi + d)^2}}\, d\xi \\ &= u(-d/c) - \frac{1}{c}\sqrt{1 - (cx + d)^2} \\ &= u(-d/c) - \frac{c}{|c|}\sqrt{c^2 - (x + d/c)^2}. \end{aligned}$$

If c is negative, we have here the upper semicircle determined by the circle

$$\left(x + \frac{d}{c}\right)^2 + \left(u(x) - u\left(-\frac{d}{c}\right)\right)^2 = c^2$$

with

$$\text{center } \left(-\frac{d}{c}, u\left(-\frac{d}{c}\right)\right) \qquad \text{and radius } |c|.$$

If $c > 0$, we have found the closed lower half of the same circle. Notice that in either case,

$$u \in C^0[-d/c - r, -d/c + r] \cap C^1(-d/c - r, -d/c + r) \backslash C^1[-d/c - r, -d/c + r].$$

Note that any semicircle in the plane may be obtained as a solution of this ODE.

The case $c = 0$ remains. In this case, we do not obtain the graph of a semicircle. The equation (3) is still valid with $c = 0$, and upon squaring we find

$$u' = \frac{d}{\sqrt{1 - d^2}}.$$

Note that for each d with $|d| < 1$, the right side determines a constant m. Furthermore, the function $m : (-1, 1) \to \mathbb{R}$ given by

$$m(d) = \frac{d}{\sqrt{1 - d^2}}$$

is monotone increasing and surjective. Therefore, the solutions in this case give precisely every affine function $u(x) = mx + b$.

In summary, the ODE in this problem (the ODE of graphs of constant curvature) gives every circle (by upper and lower halfs) and every (non-vertical) straight line graph as a solution—and exactly these.

It helps to do a problem like this if you have some picture in your mind. The more you know, the more you know.

Problem 5 *(Boas 9.1.2) Assume $A = (0, h)$ is a point on the positive y-axis (with $h > 0$) and $B = (x_0, y_0)$ is a point in the fourth quadrant with $x_0 \geq 0$ and $y_0 < 0$. Let c denote the speed of light and assume a "light particle" takes a straight line path from A to a point $p = (x, 0)$ on the x-axis moving with speed c/n_1 and the same particle continues taking a straight line path from p to B moving with speed c/n_2.*

(a) *Compute the total time for this particle to travel from A to B as a function of x.*

(b) *Find the point p on the x-axis for which the travel time from A to B is the minimum possible.*

(c) *Use your result to verify Snell's law of refraction:*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction.

Solution: The geometry described in the problem is illustrated in Figure 1.

(a) The total time of travel $T = T(x)$ is determined by applying the relation

$$\text{time of travel} = \frac{\text{distance}}{\text{rate}}$$

to each segment of travel so that

$$T = \frac{\sqrt{x^2+h^2}}{c/n_1} + \frac{\sqrt{(x_0-x)^2+y_0^2}}{c/n_2}.$$

(b) (initial comments) This problem turns out to be much more difficult than one might initially guess, so I limit myself here to some preliminary observations. In particular, I will demonstrate that, assuming $x_0 > 0$, there **exists** a **unique** value $x \in (0, x_0)$ depending smoothly on x_0, $y_0 < 0$, and the ratio $\lambda = n_1/n_2 > 0$. Certain other observations will be established for future reference.

It should be emphasized that, while the problem is difficult, it is a reasonable problem to consider, and we will show here that it has a well-defined solution. After presenting a solution for part (c), I will also explain why there should be (at least in principle) a closed form (formula) for that solution.

We note first that $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{c}{n_2} T = \lambda\sqrt{x^2+h^2} + \sqrt{(x_0-x)^2+y_0^2}$$

is a smooth function of x with

$$f'(x) = \lambda\frac{x}{\sqrt{x^2+h^2}} - \frac{x_0-x}{\sqrt{(x_0-x)^2+y_0^2}}$$

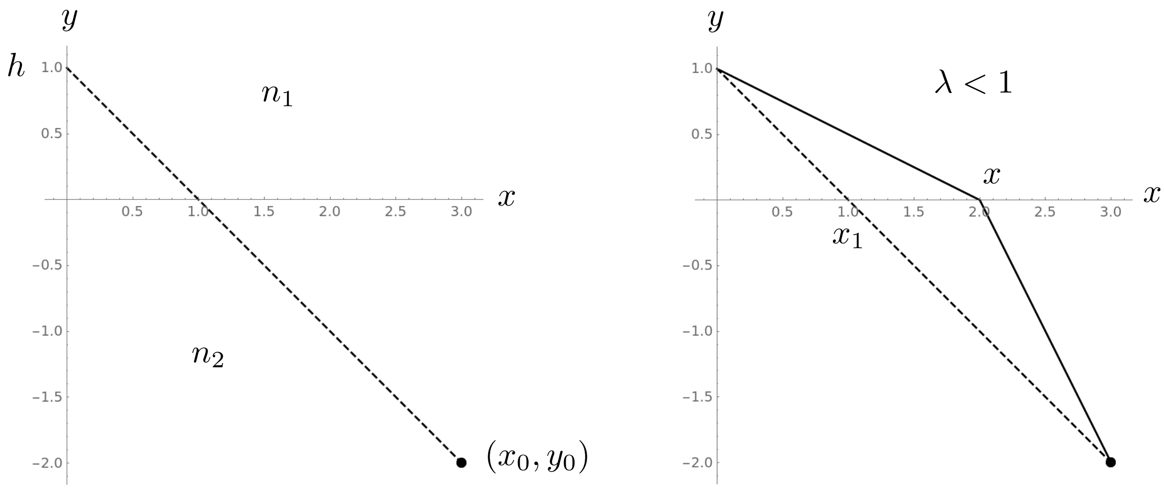


Figure 1: An origin point $(0, h)$ for light that refracts across $y = 0$ arriving at a destination point (x_0, y_0) with $x_0 > 0$ and $y_0 < 0$.

and

$$f''(x) = \lambda \frac{h^2}{(x^2 + h^2)^{3/2}} + \frac{y_0^2}{[(x_0 - x)^2 + y_0^2]^{3/2}} > 0. \tag{4}$$

The inequality (4) implies $f' : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Furthermore, $f'(0) < 0$ and $f'(x_0) > 0$. It follows that f has a unique minimum at a point $x \in (0, x_0)$. The smoothness of the minimizer $x : (0, \infty) \times (-\infty, 0) \times (0, \infty) \to \mathbb{R}$ by $x = x(x_0, y_0, \lambda)$ as a function of the three parameters x_0, y_0, and λ follows from the implicit function theorem. We henceforth denote the unique solution of the problem by x as well as the variable x indicated in Figure 1; the context should make the intended identity of this symbol clear.

The fact that $f''(x) > 0$ means that x is a, so called, **simple zero** of $f''(x) = 0$, and numerical approximation of x should be reasonably easy to obtain for any specific given values of x_0, y_0, and λ.

We define an auxiliary value $x_1 = hx_0/(h - y_0)$ giving the x-intercept of the line connecting $(0, h)$ to the point $B = (x_0, y_0)$. When $\lambda = 1$, the condition $f'(x) = 0$ implies

$$x^2[(x_0 - x)^2 + y_0^2] = (x_0 - x)^2(x^2 + h^2) \qquad \text{or} \qquad x^2 y_0^2 = h^2(x_0 - x)^2$$

so that $-xy_0 = h(x_0 - x) = hx_0 - hx$ and

$$x = x_1 = \frac{hx_0}{h - y_0}.$$

Thus, when $n_1 = n_2$ the path of minimum travel is along the straight line connecting $A = (0, h)$ and $B = (x_0, y_0)$. This is one case in which x may be easily found in terms of an explicit formula.

Holding x_0 and y_0 fixed and differentiating the defining relation

$$f'(x) = f'(x(\lambda); \lambda) = 0$$

with respect to λ, we obtain

$$f''(x)\frac{\partial x}{\partial \lambda} + \frac{x}{\sqrt{x^2 + h^2}} = 0.$$

Consequently,

$$\frac{\partial x}{\partial \lambda} = -\frac{x}{f''(x)\sqrt{x^2 + h^2}} < 0.$$

It follows that $x > x_1$ when $\lambda < 1$ and $x < x_1$ when $\lambda > 1$. This condition is illustrated on the right in Figure 1 for $\lambda < 1$ and may be interpreted to mean that the light "prefers" to travel a shorter path in the medium in which travel is slower.

The existence, uniqueness, and regularity of x extends to the case $x_0 \leq 0$ with $x = x_1 = 0$ when $x_0 = 0$ and $x < 0$ when $x_0 < 0$.

(c) Snell's law of refraction may be obtained directly, given the existence and uniqueness of $x \in [0, x_0)$ described above. We simply draw the vertical line through x as indicated in Figure 2 and observe that the condition

$$n_2 f'(x) = n_1 \frac{x}{\sqrt{x^2 + h^2}} - n_2 \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + y_0^2}} = 0$$

can be written as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

according to the geometry of the figure. This also holds trivially in the case when $x_0 = 0 = x = x_1$ for which $\theta_1 = \theta_2 = 0$.

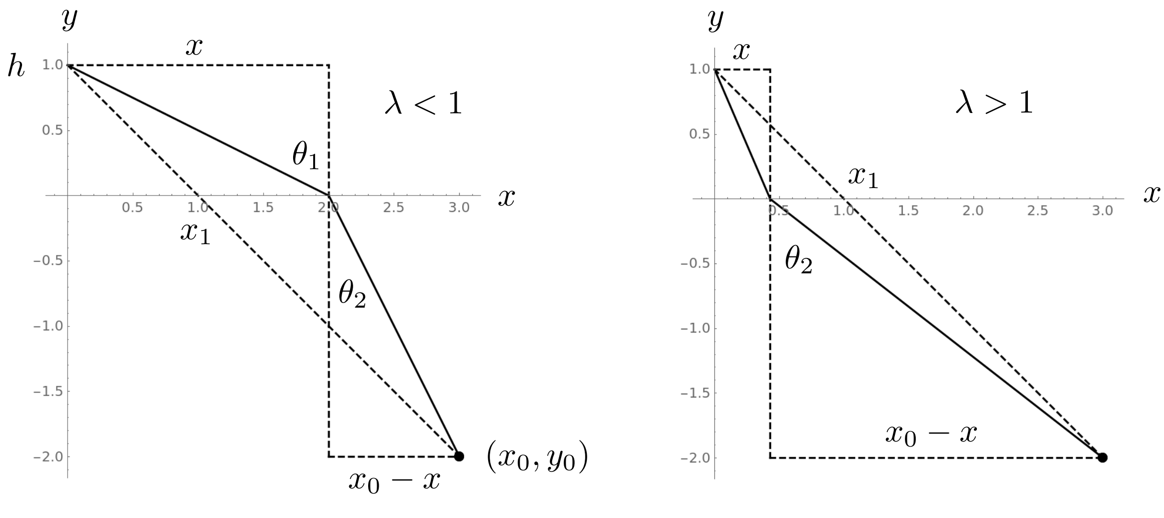


Figure 2: Snell's law.

(b) (further comments)

It may be noted that we did not consider dependence on the initial height h at which our hypothetical ray of light originated. This is because the condition defining x is essentially invariant under a scaling of the plane by $1/h$. More precisely, for each $h > 0$ the defining condition

$$f'(x) = \lambda \frac{x}{\sqrt{x^2 + h^2}} - \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + y_0^2}} = 0$$

is equivalent to

$$\lambda \frac{x/h}{\sqrt{(x/h)^2 + 1}} - \frac{x_0/h - x/h}{\sqrt{(x_0/h - x/h)^2 + (y_0/h)^2}} = 0.$$

Thus, if we can solve the problem

$$\lambda \frac{x}{\sqrt{x^2 + 1}} - \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + y_0^2}} = 0 \tag{5}$$

corresponding to the case $h = 1$, then we can solve the problem with $\tilde{A} = (0, h)$ and $\tilde{B} = (\tilde{x}_0, \tilde{y}_0)$ simply but substituting $x_0 = \tilde{x}_0/h$ and $y_0 = \tilde{y}_0/h$ in (5) and taking the solution $\tilde{x} = hx$. In view of this remark, we henceforth consider $h = 1$ and the equation (5).

If we write (5) in the form

$$\lambda \frac{x}{\sqrt{x^2+1}} = \frac{x_0 - x}{\sqrt{(x_0-x)^2+y_0^2}}$$

and square both sides, we find x satisfies

$$\lambda^2 x^2[(x_0-x)^2+y_0^2] = (x_0-x)^2(x^2+1). \tag{6}$$

Upon further rearrangement we see x satisfies the quartic polynomial equation

$$(\lambda^2-1)x^4 - 2(\lambda^2-1)x_0\,x^3 + [(\lambda^2-1)x_0^2+\lambda^2 y_0^2-1]\,x^2 + 2x_0\,x - x_0^2 = 0.$$

As described in our first discussion of part (b), when $\lambda = 1$, we have the explicit (though somewhat uninteresting) solution

$$x_1 = \frac{x_0}{1-y_0}.$$

It will be observed that in the case $\lambda = 1$, the quartic equation for x becomes

$$(y_0^2-1)x^2 + 2x_0\,x - x_0^2 = 0.$$

For $y_0 = -1$, this equation gives only the correct root $x_1 = x_0/2$, but when $y_0 \neq -1$, we find an extraneous root

$$\tilde{x}_1 = \frac{x_0}{1+y_0}$$

which satisfies $x_1 < \tilde{x}_1$ if $-1 < y_0 < 0$ and $\tilde{x}_1 < 0$ if $y_0 < -1$. In particular, the limit as y_0 tends to -1 of $\tilde{x}_1$ does not exist but

$$\lim_{y_0 \searrow -1} \tilde{x}_1 = +\infty \qquad \text{and} \qquad \lim_{y_0 \nearrow -1} \tilde{x}_1 = -\infty.$$

This tells us we have rather strong singular behavior of the (extraneous roots of the) polynomial equation for x as y_0 tends to -1 even in this simple case. In particular, we should expect to consider a number of distinct cases depending on the values of x_0, y_0, and λ. See Figure 3.

At this point, it is worth noting something about the study (and the history of the study) of quartic (and cubic and quadratic) polynomial equations in general. Actually, we are only interested in the subclass of polynomial equations with real coefficients.

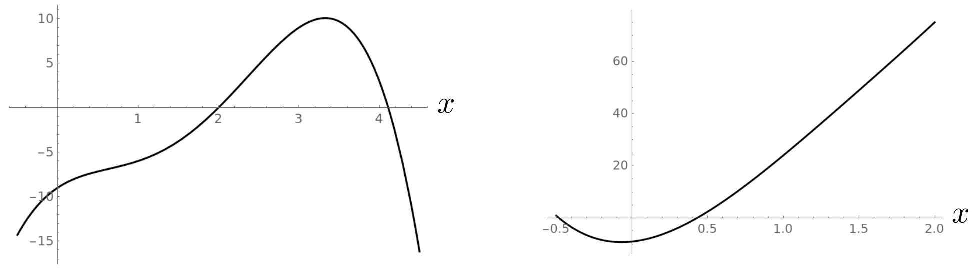


Figure 3: Here we have plotted the basic quartic polynomial for the crossing x value when $(x_0, y_0) = (3, -2)$. On the left we have taken $\lambda = 1/2$, and it can be checked explicitly that $x = 2$. On the right we have taken $\lambda = 2$, and I do not know the crossing x explicitly, but I found a numerical approximation for it. This value appears on the right in Figure 2. It may be noted that in both cases there is precisely one extraneous root $\tilde{x}$ with $\tilde{x} > x_0$ when $\lambda = 1/2$ and $\tilde{x} < 0$ when $\lambda = 2$. It seems unlikely that there are always only two roots with one extraneous root (and two complex conjugate roots), but that may indeed be the case. It also seems unlikely that the interval $(0, x_0)$ containing x will always be free of extraneous roots, but that is also true in the two examples we have here.

1. First let us note that the complex roots of such an equation, if there are any, must come in **complex conjugate pairs**. For quadratic equations, this means there are precisely two possibilities: Either both roots are real (possibly repeated) or they are nontrivially complex (and distinct). What precisely happens is, in principle, easily seen from the coefficients and from the **quadratic formula**
$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$
for the equation $x^2 + bx + c = 0$.

   There are also, in principle, known formulas for cubic and quartic equations, but the property that the nature of the roots is easily determined from the coefficients is, as far as I know, neither known to hold nor known to be unprovable.

2. To follow up on the last comment: We do know there are particular cubic and quartic equations for which the formula for the roots is easily obtained. In particular, if one real root of a cubic equation is known, then a formula for each root is easily determined. For example, we know $x = c \in \mathbb{R}$ is a

root of $x^3 - c^3 = 0$. Thus, the other two roots can be easily written down using the quadratic formula. A similar comment holds for $x^4 - c = 0$. Of course, these are very simple equations.

3. It may be that a quartic equation is known to have a real root, and one might hope there exists a simple(r) formula (than the one presently known) for the real root in such a case. This does not appear to be known, but as far as I know it may still be hoped that such a formula can be discovered at least for some particular special equations.
4. The immediate question arises then: Is our quartic equation special in any particular sense?
5. We have already seen that even in a simple degenerate case, the polynomial under consideration can have extraneous roots smaller than or larger than the desired crossing point x. It would be nice if we at least knew x was, say, the unique largest (and positive) real root of the polynomial equation. I am going to give a construction in the case $\lambda < 1$ leading to a polynomial equation with something like this property—essentially with this property in fact. Hopefully, the construction will appear justified after the details are given.
6. In the course of the discussion, I will attempt to treat the case $\lambda < 1$ in full generality to the extent I am able to do so.
7. I will also consider a specific example $(x_0, y_0) = (3, -2)$ and $\lambda = 1/2$ for which I will attempt a full solution.

Returning to our quartic equation (6) obtaind by squaring, we can rearrange the terms in a different way:

$$\lambda^2 x^2 y_0^2 = [(1 - \lambda^2)x^2 + 1](x_0 - x)^2.$$

Notice that the quantity $(1-\lambda^2)x^2+1$ is greater than 1. Therefore, the principal (real positive) square root

$$\mu = \mu(x) = \sqrt{(1 - \lambda^2)x^2 + 1} > 1$$

is well-defined. Taking a square root in the defining relation, we obtain the relation

$$-\lambda x y_0 = \mu(x_0 - x).$$

That is,

$$x = \frac{x_0 \mu}{\mu - \lambda y_0}. \tag{7}$$

Note that if we can find $\mu^2 > 1$, then we can find x:

$$x = \sqrt{\frac{\mu^2 - 1}{1 - \lambda^2}}.$$

In particular, making this substitution on the left in (7) and squaring we find

$$(\mu - \lambda y_0)^2(\mu^2 - 1) = (1 - \lambda^2)x_0^2\mu^2$$

Simplifying this relation, we obtain

$$(\mu^2 - 2\lambda y_0 \mu + \lambda^2 y_0^2)(\mu^2 - 1) = (1 - \lambda^2)x_0^2\mu^2$$

and

$$\mu^4 - 2\lambda y_0\, \mu^3 + (\lambda^2 x_0^2 - x_0^2 + \lambda^2 y_0^2 - 1)\, \mu^2 + 2\lambda y_0\, \mu - \lambda^2 y_0^2 = 0.$$

Thus, μ satisfies a quartic equation. We will show, moreover, that this equation has a unique positive root. We will show the root is simple, i.e., that the derivative of the polynomial is nonzero (in fact postive) at the root, and a number of other properties.

Let

$$q(\mu) = \sum_{j=0}^{4} q_j \mu^j = \mu^4 - 2\lambda y_0\, \mu^3 + (\lambda^2 x_0^2 - x_0^2 + \lambda^2 y_0^2 - 1)\, \mu^2 + 2\lambda y_0\, \mu - \lambda^2 y_0^2$$

with the most interesting coefficient

$$\begin{aligned} q_2 &= \lambda^2 x_0^2 - x_0^2 + \lambda^2 y_0^2 - 1 \\ &= \lambda^2(x_0^2 + y_0^2) - x_0^2 - 1 \\ &= -(1 - \lambda^2)x_0^2 + \lambda^2 y_0^2 - 1 \\ &= \lambda^2 y_0^2 - m(x_0, \lambda) \end{aligned}$$

where $m = m(x, \lambda) = (1 - \lambda^2)x^2 + 1 > 1$ is a function we have seen before in the sense that

$$\mu = \sqrt{m(x, \lambda)}.$$

The other coefficients have known signs with

$$\begin{aligned} q(0) &= q_0 = -\lambda^2 y_0^2 < 0, \\ q'(0) &= q_1 = 2\lambda y_0 < 0, \\ q'''(0) &= 6q_3 = -12\lambda y_0 > 0. \end{aligned}$$

The quantity $q''(0) = 2q_2(0)$ may be either positive or negative, but we can classify the behavior in terms of the position of the point (x_0, y_0) in the fourth quadrant. In fact, the third expression for q_2 given above indicates that $q_2 = q_2(x_0, y_0)$ vanishes precisely along the branch of the hyperbola

$$-\frac{x_0^2}{1/(1-\lambda^2)} + \frac{y_0^2}{1/\lambda^2} = 1$$

with vertex at $(0, -1/\lambda)$ as indicated on the left in Figure 4.

In view of the signs of the coefficients indicated above, we note that

$$q''(\mu) = 12\mu^2 - 12\lambda y_0 \mu + 2q_2 = 2\left[6\left(\mu - \frac{\lambda y_0}{2}\right)^2 - \frac{3\lambda^2 y_0^2}{2} + q_2\right]$$

has a minimum value

$$q''\left(\frac{\lambda y_0}{2}\right) = 2q_2 - 3\lambda^2 y_0^2 = -2(1-\lambda^2)x_0^2 - \lambda^2 y_0^2 - 2 < 0$$

at $\mu = \lambda y_0/2$. Therefore, q'' has precisely two sign changes at precisely the values

$$\mu_\pm = \frac{\lambda y_0}{2} \pm \sqrt{\frac{\lambda^2 y_0^2}{4} - \frac{q_2}{6}}.$$

At most one of these values of $\mu_\pm$ is positive. More precisely, if $q''(0) = 2q_2 > 0$, then both sign changes of q'' correspond to negative values of mu and q is convex for $\mu \ge 0$. If $q_2 = 0$, then q'' has a sign change at $\mu_+ = 0$, but q is still (strictly) convex for $\mu > 0$. If $q_2 < 0$, then there is some interval $0 < \mu < \mu_+$ on which $q'' < 0$, but $q''(\mu) > 0$ for $\mu > \mu_+$. This last observation is illustrated on the right in Figure 4.

Recalling that $q(0) < 0$ and $q'(0) < 0$, if $q''(\mu) > 0$ for $\mu > 0$, then there can be (and there is) precisely one positive root μ of the equation $q(\mu) = 0$. Since we know the desired root μ corresponding to the desired crossing x satisfies

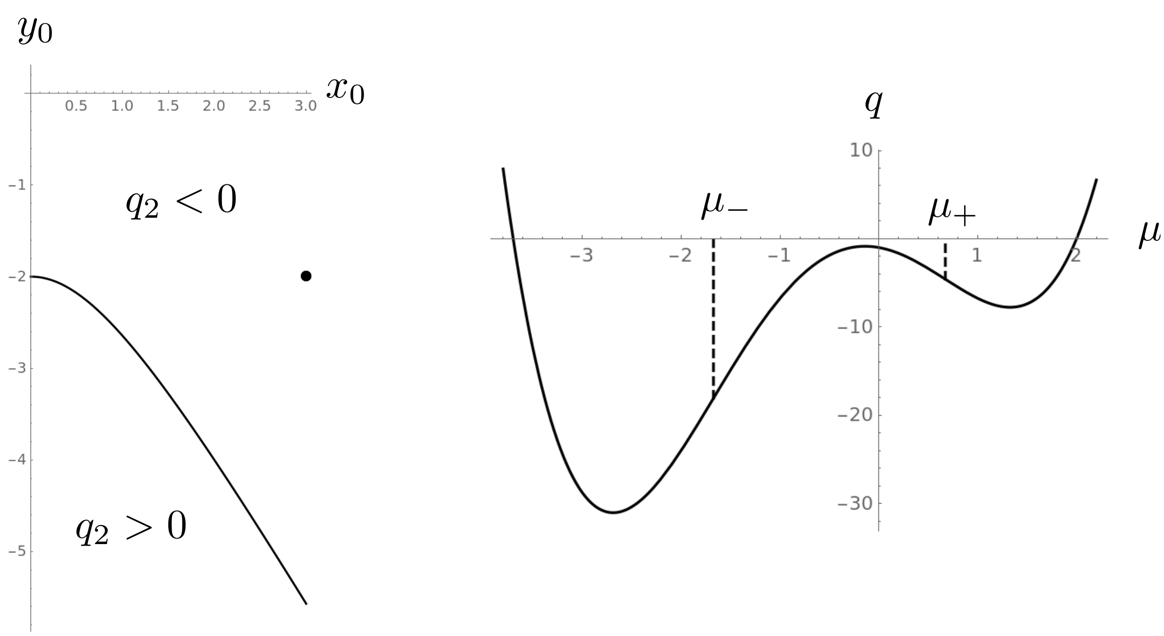
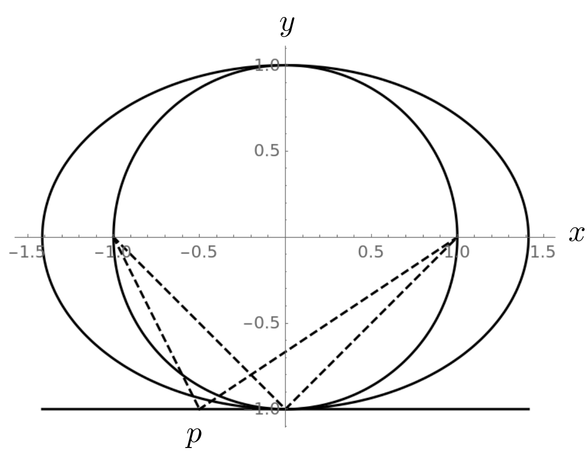


Figure 4: On the left we have plotted the curve where the coefficient q_2 changes sign in the (x_0, y_0) plane when $\lambda = 1/2$. It will be noted that our chosen point $(3, -2)$ lies in the region where $q_2 < 0$ corresponding to a negative second derivative for q at $x = 0$. This is clearly seen on the right where we have plotted the polynomial q for these choices ($\lambda = 1/2$ and $(x_0, y_0) = (3, -2)$).

$q(\mu) = 0$ and $\mu > 1$, we know the unique positive root is also this root and satisfies $\mu > 1$.

Similarly, if $q''(0) < 0$, then q' is decreasing and negative for $0 \leq \mu < \mu_+$. In particular, $q(\mu) < 0$ for $0 \leq \mu \leq \mu_+$ with $q(\mu_+) < q(0) < 0$ and $q'(\mu_+) < q'(0) < 0$. Furthermore, q is convex on $\mu_+ < \mu$. Again it follows that there can be (and there is) precisely one positive root μ of the equation $q(\mu) = 0$. This root satisfies $\mu > \mu_+$ and is also the desired root satisfying $\mu > 1$ by the reasoning above.

Problem 6 *(Boas 9.1.1,3) The figure below shows the ellipse $x^2/2 + y^2 = 1$ with semi-axes of lengths $\sqrt{2}$ and 1 and focal points $(\pm 1, 0)$. Also shown are the inscribed circle $x^2 + y^2 = 1$ and the tangent line $y = -1$ at $(0, -1)$. Each of these three curves may be considered as the top view of a reflecting wall, and a light ray emitted from $(-1, 0)$ is shown reflecting off each of these walls at $(-1, 0)$ and subsequently reaching $(1, 0)$. The angle of reflection is equal to the angle of incidence for this path in accordance with Hero's law of reflection.*



(a) *Assume a ray of light travels with speed c/n (where c is the speed of light in a vacuum and $n > 1$ is a constant). Consider all paths along which light may travel from the point $(-1, 0)$ along a straight line to a point $p = (x, -1)$ and then travel along a straight line from p to $(1, 0)$. We can say such a path models the light "bouncing" off the line at p. Show the path bouncing at $(0, -1)$ is the path of least travel time among all paths that model light bouncing off the flat wall $y = -1$ at points $p = (x, -1)$.*

(b) *Show all piecewise straight line paths starting at* $(-1, 0)$ *and reflecting off the ellipse* $x^2/2 + y^2 = 1$ *at points* $p = (x, y)$ *and going (straight) to* $(1, 0)$ *have the same travel time and the same angles of incidence and reflection.*

(c) *Show all unions of two straight line segments with the first connecting* $(-1, 0)$ *to a point* p *on the circle and the second connecting* p *to* $(1, 0)$ *have travel times* **strictly less** *than the actual path (of reflection through* $(0, -1)$*) unless* $p = (0, \pm 1)$.

Problem 7 *(Boas §9.2) Let* $u \in C^1[a, b]$.

(a) *Given a partition* $\mathcal{P} = \{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ *Consider the (Riemann) sum*

$$\sum_{j=1}^{k} \sqrt{[x_j - x_{j-1}]^2 + [u(x_j) - u(x_{j-1})]^2}.$$

Draw a picture showing the geometric meaning of this sum, and use the mean value theorem to write this sum as a Riemann sum in the form

$$\sum_{j=1}^{k} F(u'(x_j^*))\,(x_j - x_{j-1})$$

for some evaluation points $x_1^*, x_2^*, \ldots, x_k^*$.

(b) *Take the limit*

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{k} \sqrt{[x_j - x_{j-1}]^2 + [u(x_j) - u(x_{j-1})]^2}$$

where $\|\mathcal{P}\| = \max_j (x_j - x_{j-1})$ *to obtain a functional* $\mathcal{L} : C^1[a, b] \to \mathbb{R}$ *of the form*

$$\mathcal{L}[u] = \int_a^b F(u'(x))\,dx.$$

(c) *Consider*

$$M = \{w \in C^1[a, b] : \mathcal{L}[w] \le \mathcal{L}[u] \qquad \text{for all } u \in C^1[a, b]\}.$$

Characterize the set M. *Prove your assertion and determine if* M *is a vector subspace of* $C^1[a, b]$. M *is called the* **set of minimizers** *for* $\mathcal{L}$.

(d) *Consider* $\mathcal{A} = \{u \in C^1[a,b] : u(a) = 0 \text{ and } u(b) = 1\}$*. Find the set of minimizers of the restriction of* $\mathcal{L}$ *to* $\mathcal{A}$*. Can you prove your assertion?*

Problem 8 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be two distinct points fixed in the plane* $\mathbb{R}^2$*. The set of* C^1 **paths** *connecting* $\mathbf{p}$ *to* $\mathbf{q}$ *is*

$$\mathcal{A} = \{\gamma \in C^1([0,1] \to \mathbb{R}^2) : \gamma(0) = \mathbf{p} \text{ and } \gamma(1) = \mathbf{q}\}.$$

(a) *Find all circular arcs of a fixed radius in* $\mathcal{A}$*.*

(b) *Write down a functional* $\mathcal{L} : \mathcal{A} \to \mathbb{R}$ *for which* $\mathcal{L}[\gamma]$ *is the length of the path* γ*.*

(c) *Compute the first variation of* $\mathcal{L}$*. (Hint: It may be helpful to write down an appropriate set* $\mathcal{V}$ *of admissible perturbations for this problem.)*

Problem 9 *Let* $L > |\mathbf{q} - \mathbf{p}|$ *where* $\mathbf{p} = (a, y_1)$ *and* $\mathbf{q} = (b, y_2)$ *are fixed points in the plane* $\mathbb{R}^2$ *with* $a < b$*. A chain lies in the plane taking a certain shape modeled by the graph of a function* $u \in C^1[a,b]$ *with* $u(a) = y_1$ *and* $u(b) = y_2$*.*

(a) *Imagine the chain consists of "links" modeled by*

$$L_j = \{(x, u(x)) : x_{j-1} \le x \le x_j\} \qquad \text{for } j = 1, 2, \ldots, k \tag{8}$$

where $\mathcal{P} = \{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ *is a partition of* $[a,b]$*. Imagine further that the chain is constructed by moving each link* L_j *vertically from the position*

$$\{(x, u(x) - u(x_j)) : x_{j-1} \le x \le x_j\}$$

to the position (8) through a downward gravitational potential field $-g(0,1)$*. Write down a Riemann sum giving the total work (i.e., energy) required to construct the chain in this way. Hint: Assume a uniform linear density* ρ *along the chain so that any length* ℓ *of this (kind of) chain has mass* $\rho\ell$*.*

(b) *Take a limit of your approximate potential energy/Riemann sum to obtain a potential energy function* $\mathcal{E} : \mathcal{A} \to \mathbb{R}$ *assigning a potential energy to each model chain shape. Hint: Writing down the definition of the admissible class* $\mathcal{A}$ *which is the domain of* $\mathcal{E}$ *is part of the problem.*

(c) *Show* $\mathcal{E}$ *is not bounded below on* $\mathcal{A}$*.*

(d) *Introduce an appropriate* **constraint** *within the admissible class $\mathcal{A}$ according to which there is some hope to minimize $\mathcal{E}$. Hint: Look at the very first hypothesis in the statement of this problem, and then use Problem 7 above. Your answer may be given in terms of an appropriate subset $\mathcal{A}_L$ of $\mathcal{A}$ determined by the constraint.*

Note: You should not expect to be able to carry out the mathematical details of minimizing $\mathcal{E}$ on $\mathcal{A}$ subject to the constraint you gave in part (d), but you should have a strong physical intuition that a minimizer for this constrained problem should exist. Soon you should be able to find it.

Problem 10 *The previous problem involved minimizing a real valued function(al) subject to a constraint. Here is a finite dimensional version of this kind of problem: Minimize the value of $u(x, y) = x^2 + y^2$ on $\mathbb{R}^2$ subject to the constraint $x^2/2 + y^2 = 1$. See Boas §4.9 and §9.6.*