Lecture 1: Ordinary Differential Equations and The Calculus of Variations Assignment Problems Due Monday February 8, 2021

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Problem 1 *(Boas 8.6.1) Find the general solution of* $y'' - 4y = 10$*. Solve the IVP*

$$\begin{cases} y'' - 4y = 10 \\ y(1) = -3, \;\; y'(1) = -2. \end{cases}$$

Use mathematical software to find a numerical approximation of the solution of the IVP. (Also plot your solution to see the two match.)

Problem 2 *(Boas 8.6.28) Consider the following ordinary differential operators on complex valued functions of a real variable:*

$$\frac{d}{dt} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \frac{d}{dt} u = u'$$

and

$$\mathrm{id} : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad \mathrm{id}\, u = u.$$

(a) *Expand the linear constant coefficient operator*

$$L : C^\infty(\mathbb{R} \to \mathbb{C}) \to C^\infty(\mathbb{R} \to \mathbb{C}) \qquad \text{by} \qquad Lu = \left(\frac{d}{dt} - a\,\mathrm{id} \right) \left(\frac{d}{dt} - b\,\mathrm{id} \right) u$$

where a *and* b *are complex numbers to obtain an expression of the form* $Lu = u'' + pu' + qu$ *for complex numbers* p *and* q*.*

(b) *Find the general solution of* $Lu = ke^{ct}$ *where* k *and* c *are complex numbers by solving* $y' - ay = ke^{ct}$ *first and then solving* $u' - bu = y$ *(as linear first order ODEs) in the three cases:*

- **(i)** $c \neq a$ *and* $c \neq b$.
- **(ii)** $a \neq b$ *and* $c = a$.
- **(iii)** $a = b = c$.

Problem 3 *(Boas 8.7.5) The shape of a hanging chain is modeled by solutions of*

$$(y'')^2 = k^2[1 + (y')^2].$$

Find the general solution of this (nonlinear) ODE.

Problem 4 *(Boas 8.7.6) The* **signed curvature** *of the graph of a function* $u \in C^2[a,b]$ *at the point* $(x, u(x))$ *is defined to be the derivative*

$$k = \frac{d\psi}{ds}$$

with respect to arclength

$$s = \int_a^x \sqrt{1 + [u'(\xi)]^2}\, d\xi$$

of the inclination angle ψ *defined by*

$$(\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1 + [u'(x)]^2}}, \frac{u'(x)}{\sqrt{1 + [u'(x)]^2}} \right).$$

(a) *Find the curvature of the graph of* $u(x) = \sqrt{r^2 - x^2}$ *for* $|x| < r$.

(b) *Find the curvature of the graph of* $u(x) = -\sqrt{r^2 - x^2}$ *for* $|x| < r$.

(c) *Show the curvature is given in general by*

$$k = \frac{u''}{(1 + [u'(x)]^2)^{3/2}}.$$

(d) *Solve the ODE*

$$\frac{u''}{(1 + [u'(x)]^2)^{3/2}} = c$$

where c *is a (real) constant.*

Problem 5 *(Boas 9.1.2) Assume $A = (0, h)$ is a point on the positive y-axis (with $h > 0$) and $B = (x_0, y_0)$ is a point in the fourth quadrant with $x_0 \geq 0$ and $y_0 < 0$. Let c denote the speed of light and assume a "light particle" takes a straight line path from A to a point $p = (x, 0)$ on the x-axis moving with speed c/n_1 and the same particle continues taking a straight line path from p to B moving with speed c/n_2.*

(a) *Compute the total time for this particle to travel from A to B as a function of x.*

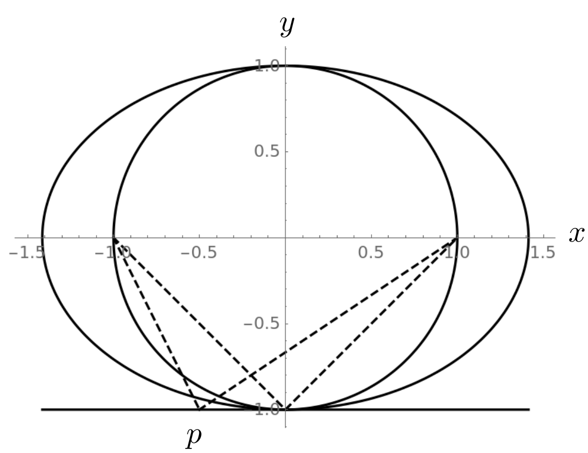
(b) *Find the point p on the x-axis for which the travel time from A to B is the minumum possible.*

(c) *Use your result to verify Snell's law of refraction:*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction.

Problem 6 *(Boas 9.1.1,3) The figure below shows the ellipse $x^2/2 + y^2 = 1$ with semi-axes of lengths $\sqrt{2}$ and 1 and focal points $(\pm 1, 0)$. Also shown are the inscribed circle $x^2 + y^2 = 1$ and the tangent line $y = -1$ at $(0, -1)$. Each of these three curves may be considered as the top view of a reflecting wall, and a light ray emitted from $(-1, 0)$ is shown reflecting off each of these walls at $(-1, 0)$ and subsequently reaching $(1, 0)$. The angle of reflection is equal to the angle of incidence for this path in accordance with Hero's law of reflection.*



(a) *Assume a ray of light travels with speed c/n (where c is the speed of light in a vacuum and $n > 1$ is a constant). Consider all paths along which light may* *travel from the point $(-1,0)$ along a straight line to a point $p=(x,-1)$ and then travel along a straight line from p to $(1,0)$. We can say such a path models the light "bouncing" off the line at p. Show the path bouncing at $(0,-1)$ is the path of least travel time among all paths that model light bouncing off the flat wall $y=-1$ at points $p=(x,-1)$.*

(b) *Show all piecewise straight line paths starting at $(-1,0)$ and reflecting off the ellipse $x^2/2+y^2=1$ at points $p=(x,y)$ and going (straight) to $(1,0)$ have the same travel time and the same angles of incidence and reflection.*

(c) *Show all unions of two straight line segments with the first connecting $(-1,0)$ to a point p on the circle and the second connecting p to $(1,0)$ have travel times* **strictly less** *than the actual path (of reflection through $(0,-1)$) unless $p=(0,\pm 1)$.*

Problem 7 *(Boas §9.2) Let $u \in C^1[a,b]$.*

(a) *Given a partition $\mathcal{P}=\{a=x_0<x_1<x_2<\cdots<x_k=b\}$ Consider the (Riemann) sum*

$$\sum_{j=1}^{k}\sqrt{[x_j-x_{j-1}]^2+[u(x_j)-u(x_{j-1})]^2}.$$

Draw a picture showing the geometric meaning of this sum, and use the mean value theorem to write this sum as a Riemann sum in the form

$$\sum_{j=1}^{k}F(u'(x_j^*))\,(x_j-x_{j-1})$$

for some evaluation points $x_1^*, x_2^*, \ldots, x_k^*$.

(b) *Take the limit*

$$\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{k}\sqrt{[x_j-x_{j-1}]^2+[u(x_j)-u(x_{j-1})]^2}$$

where $\|\mathcal{P}\|=\max_j(x_j-x_{j-1})$ to obtain a functional $\mathcal{L}: C^1[a,b]\to\mathbb{R}$ of the form

$$\mathcal{L}[u]=\int_a^b F(u'(x))\,dx.$$

(c) *Consider*

$$M = \{w \in C^1[a,b] : \mathcal{L}[w] \le \mathcal{L}[u] \qquad \text{for all } u \in C^1[a,b]\}.$$

Characterize the set M. Prove your assertion and determine if M is a vector subspace of $C^1[a,b]$. M is called the **set of minimizers** *for $\mathcal{L}$.*

(d) *Consider $\mathcal{A} = \{u \in C^1[a,b] : u(a) = 0 \text{ and } u(b) = 1\}$. Find the set of minimizers of the restriction of $\mathcal{L}$ to $\mathcal{A}$. Can you prove your assertion?*

Problem 8 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be two distinct points fixed in the plane $\mathbb{R}^2$. The set of C^1* **paths** *connecting* $\mathbf{p}$ *to* $\mathbf{q}$ *is*

$$\mathcal{A} = \{\gamma \in C^1([0,1] \to \mathbb{R}^2) : \gamma(0) = \mathbf{p} \text{ and } \gamma(1) = \mathbf{q}\}.$$

(a) *Find all circular arcs of a fixed radius in $\mathcal{A}$.*

(b) *Write down a functional $\mathcal{L} : \mathcal{A} \to \mathbb{R}$ for which $\mathcal{L}[\gamma]$ is the length of the path γ.*

(c) *Compute the first variation of $\mathcal{L}$. (Hint: It may be helpful to write down an appropriate set $\mathcal{V}$ of admissible perterbations for this problem.)*

Problem 9 *Let $L > |\mathbf{q} - \mathbf{p}|$ where $\mathbf{p} = (a, y_1)$ and $\mathbf{q} = (b, y_2)$ are fixed points in the plane $\mathbb{R}^2$ with $a < b$. A chain lies in the plane taking a certain shape modeled by the graph of a function $u \in C^1[a,b]$ with $u(a) = y_1$ and $u(b) = y_2$.*

(a) *Imagine the chain consists of "links" modeled by*

$$L_j = \{(x, u(x)) : x_{j-1} \le x \le x_j\} \qquad \text{for } j = 1, 2, \ldots, k \tag{1}$$

where $\mathcal{P} = \{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ is a partition of $[a,b]$. Imagine further that the chain is constructed by moving each link L_j vertically from the position

$$\{(x, u(x) - u(x_j)) : x_{j-1} \le x \le x_j\}$$

to the position (1) through a downward gravitational potential field $-g(0,1)$. Write down a Riemann sum giving the total work (i.e., energy) required to construct the chain in this way. Hint: Assume a uniform linear density ρ along the chain so that any length ℓ of this (kind of) chain has mass $\rho\ell$.

(b) *Take a limit of your approximate potential energy/Riemann sum to obtain a potential energy function* $\mathcal{E} : \mathcal{A} \to \mathbb{R}$ *assigning a potential energy to each model chain shape. Hint: Writing down the definition of the admissible class* $\mathcal{A}$ *which is the domain of* $\mathcal{E}$ *is part of the problem.*

(c) *Show* $\mathcal{E}$ *is not bounded below on* $\mathcal{A}$*.*

(d) *Introduce an appropriate* **constraint** *within the admissible class* $\mathcal{A}$ *according to which there is some hope to minimize* $\mathcal{E}$*. Hint: Look at the very first hypothesis in the statement of this problem, and then use Problem 7 above. Your answer may be given in terms of an appropriate subset* $\mathcal{A}_L$ *of* $\mathcal{A}$ *determined by the constraint.*

Note: You should not expect to be able to carry out the mathematical details of minimizing $\mathcal{E}$ *on* $\mathcal{A}$ *subject to the constraint you gave in part (d), but you should have a strong physical intuition that a minimizer for this constrained problem should exist. Soon you should be able to find it.*

Problem 10 *The previous problem involved minimizing a real valued function(al) subject to a constraint. Here is a finite dimensional version of this kind of problem: Minimize the value of* $u(x, y) = x^2 + y^2$ *on* $\mathbb{R}^2$ *subject to the constraint* $x^2/2 + y^2 = 1$*. See Boas §4.9 and §9.6.*