Math 6702, Assignment 2 = Exam 1

Introduction

1. (Exercise 8 in the notes "Introduction" from 2020) What is the first order system equivalent to the ODE
$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x)?$$
Fully justify your answer.

2. (Exercise 24 in the notes "Introduction" from 2020) Find a system of first order equations equivalent to the hyperbolic PDE
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

§4.2 Power Series

3. (Boas 4.2.2,5) Find the power series expansions for
   (a) $\cos(x+y)$ and
   (b) $\sqrt{1+xy}$.

4. The **Taylor expansion** of a function $f \in C^\infty(\mathbb{R})$ at $x_0 \in \mathbb{R}$ is given by
$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j. \tag{1}$$
Here $f^{(j)}$ denotes the j-th (ordinary) derivative of f as usual:
$$f^{(j)} = \frac{d^j f}{dx^j}.$$
A function $f \in C^\infty(\mathbb{R})$ is said to be **real analytic** in the interval $I = (x_0 - r, x_0 + r)$ if the series in (1) converges for each $x \in I$ and
$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j.$$
The set of real analytic functions is denoted by C^ω. Verify that $\cos x$ is real analytic on $\mathbb{R}$, i.e., $\cos \in C^\omega(\mathbb{R})$.

5. Find a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^\infty(\mathbb{R}) \backslash C^\omega(\mathbb{R})$. Hint: Take $x_0 = 0$ and $f(x) \equiv 0$ for all $x \le 0$. Then (try to) define $f(x)$ for $x > 0$ so that all the derivatives $f^{(j)}(0)$ are zero, but the values of $f(x)$ for $x > 0$ are nonzero. This is a pretty hard problem if you've never seen such a function before.

6. The **Taylor expansion** of a function $u \in C^\infty(U)$ at $\mathbf{x}_0 \in U \subset \mathbb{R}^n$ is given by

$$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta. \tag{2}$$

There are a lot of things in this expansion formula which are probably new to you. Don't freak out. First, just compare (2) to (1) and observe that these two formulas are the "same" or at least sort of the same, so (on the face of it) this is a pretty cool formula, if it has some sensible meaning—and it does. The exercise will lead you through what it means.

(a) In this expansion formula $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a **multi-index**, which simply means

$$\beta \in \mathbb{N}^n = \{(m_1, \ldots, m_n) : m_1, \ldots, m_n \in \mathbb{N}\} \qquad \text{where} \qquad \mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The derivative $D^\beta u$ denotes the partial derivative taken β_j times with respect to x_j for each $j = 1, 2, \ldots, n$:

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}.$$

The "length" of a multi-index β is defined by

$$|\beta| = \sum_{j=1}^{n} \beta_j.$$

Find all the multi-indices $\beta \in \mathbb{N}^3$ with $|\beta| = 2$.

(b) Write down all the second partials of a function $u : \mathbb{R}^3 \to \mathbb{R}$ in terms of multi-indices. Your answers should look like this:

$$D^{(2,0,0)} u = \frac{\partial^2 u}{\partial x^2}$$

and you should get five more for a total of six.

(c) Now let's back up a dimension to $\mathbb{R}^2$. The expansion for $f(x, y)$ given by Boas on page 192 has second order terms

$$\frac{1}{2!} \left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2 f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right].$$

The corresponding second order terms in (2) are

$$\sum_{|\beta|=2} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

where $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{x} = (x, y)$. To see that these are the same, you need to know the definition of the **factorial** of a multi-index, and you need to know how to take **multi-index powers** of a vector variable. Here are the definitions for $\beta \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$\beta! = \beta_1!\beta_2! \cdots \beta_n!.$$
$$\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}.$$

Show that the second order terms given by Boas for a function of two variables are the same ones you get from the formula given in (2) when $n = 2$.

7. Given an open set $U \subset \mathbb{R}^n$, a function $u \in C^\infty(U)$ is said to be **real analytic** if for each $\mathbf{x}_0 \in U$, there exists some $r > 0$ such that the series in (2) converges for each $\mathbf{x} \in B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < r\}$ and

$$u(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

for $\mathbf{x} \in B_r(\mathbf{x}_0) \cap U$. The set of **real analytic functions** on an open set $U \subset \mathbb{R}^n$ is denoted by $C^\omega(U)$. Find a function $u \in C^\infty(\mathbb{R}^n) \backslash C^\omega(\mathbb{R}^n)$.

Remark on notation: It is usual to denote the center of expansion of a power series in one variable by x_0 as in (1). For comparison of (2) to (1), we have used $\mathbf{x}_0$ as the (vector) center of expansion in the multivariable expansion. This causes a certain inconvenience when writing down the coordinates in higher dimensions. For $n = 2$ as in part (c) of problem 6, one can use $\mathbf{x}_0 = (x_0, y_0)$, and this approach can work for $n = 3$ as well with $\mathbf{x}_0 = (x_0, y_0, z_0)$. For general n, however, one usually resorts to something unpleasant like

$$\mathbf{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0).$$

To further understand the unpleasantness of this expression for the coordinates, you may write out the multi-index power $\mathbf{x}_0^\beta$. My preferred alternative is to replace $\mathbf{x}_0$ with $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, though some continuity of notation is lost between (1) and (2).

8. Repeat Boas' Problem 4.2.2 (given above as Problem 3) using the multi-index Taylor expansion formula.

Calculus of Variations

9. (Boas 9.2.1) Let a, b, c, and d be fixed **positive** real numbers with $0 < a < b$ and set

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = c,\ u(b) = d\}.$$

Consider the functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{F}[u] = \int_a^b \sqrt{x(1 + u'(x)^2}\, dx.$$

(a) Compute the first variation $\delta\mathcal{F}_u : C_c^\infty(a, b) \to \mathbb{R}$.

(b) Determine the conditions under which there exist functions $u \in C^2(a,b) \cap \mathcal{A}$ for which $\delta\mathcal{F}_u \equiv 0$. Hint: Consider possibilities for the ratio d/c along with the quantity $v(x) = \sin\psi = u'/\sqrt{1+u'^2}$ (the sine of the inclination angle).

(c) Assuming the conditions you determined in part (b) for the existence of a C^2 weak extremal in $\mathcal{A}$, find all C^2 weak extremals.

(d) Take specific values for a, b, c, and d satisfying the conditions you found in part (b). Compare the functional values obtained for the C^2 weak extremals from part (c) (applied with these specific values for a, b, c, and d) to other values $\mathcal{F}[u]$ obtained from other admissible functions.

Solution:

(a)

$$\delta\mathcal{F}_u[\phi] = \int_a^b \sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\phi'\,dx.$$

If $u \in C^2(a,b)$, then we can integrate by parts to write the first variation in the form

$$\delta\mathcal{F}_u[\phi] = \sqrt{b}\frac{u'(b)}{\sqrt{1+u'(b)^2}}\phi(b) - \sqrt{a}\frac{u'(a)}{\sqrt{1+u'(a)^2}}\phi' - \int_a^b \left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)\phi\,dx.$$

If we restrict to $\phi \in C_c^\infty(a,b)$, then the boundary terms vanish, and we have

$$\delta\mathcal{F}_u[\phi] = -\int_a^b \left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)'\phi\,dx.$$

(b) This is a little more complicated than the hint suggests. The Euler-Lagrange equation is

$$\left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)' = 0.$$

This means that if we have a solution then there must be a constant α such that

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}} = \frac{\alpha}{\sqrt{x}}. \tag{3}$$

The first observation is that the right side cannot change signs. This means any solution is either strictly increasing (if $\alpha > 0$) or strictly decreasing (if $\alpha < 0$). There is also the possibility that $\alpha = 0$. If $\alpha = 0$, then we must have $u' \equiv 0$ which means u is constant and to get a C^2 solution we must have $c = u(a) = u(b) = d$. So this is one condition where we do get a solution:

If $c = d$, then $u(x) \equiv c$ is a solution,

and this is the unique C^2 extremal in this case.

We also know that if $c < d$ (i.e., $c/d < 1$, then the solution (if there is one) must be increasing and $\alpha > 0$. Similarly, if $d < c$, then the only possibility is $\alpha < 0$. Thus, we have two cases to consider.

Let's consider the case $c < d$. Then we can assume $\alpha > 0$.

The function

$$\sin\psi(x) = \frac{\alpha}{\sqrt{x}}$$

is well-defined, smooth, positive, and decreasing on $[a, b]$ with maximum value

$$0 < \frac{\alpha}{\sqrt{a}} = \sin\psi(a) < 1.$$

Therefore, we must have $0 < \alpha < \sqrt{a}$. Again, if we do have a solution, then we should be able to integrate (3) as follows: We first rearrange the equation as

$$u' = \frac{\alpha/\sqrt{x}}{\sqrt{1-\alpha^2/x}} = \frac{\alpha}{\sqrt{x-\alpha^2}} = 2\alpha\frac{d}{dx}\sqrt{x-\alpha^2}.$$

Then integration gives

$$u(x) = u(a) + 2\alpha\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right] = c + 2\alpha\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right].$$

In particular the second boundary condition requires $u(b) = d$ or

$$f(\alpha) = \alpha\left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right] = \frac{d-c}{2} > 0.$$

The function $f : [0, \sqrt{a}] \to \mathbb{R}$ is smooth with

$$f(0) = 0, \quad f(\sqrt{a}) = \sqrt{a}\sqrt{b-a} > 0,$$

and

$$f'(\alpha) = \left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right]\left(1 + \frac{\alpha^2}{\sqrt{(b-\alpha^2)(a-\alpha^2)}}\right) > 0.$$

In particular, f is increasing, and the equation $f(\alpha) = (d-c)/2$ will have a unique positive solution α with $0 < \alpha < \sqrt{a}$ if and only if

$$\frac{d-c}{2} < f(\sqrt{a}) = \sqrt{a}\sqrt{b-a}.$$

Thus, the condition

$$\frac{d-c}{2} < \sqrt{a}\sqrt{b-a}$$

is required for there to exist a C^2 extremal when $c < d$.

If $c > d$, then we must have $\alpha < 0$. In this case, $\sin\psi(x)$ is negative and increasing with minimum value $\sin\psi(a) = \alpha/\sqrt{a} < 0$. We must have $\alpha/\sqrt{a} \geq -1$, so

$$-\sqrt{a} < \alpha < 0.$$

The integration proceeds in the same way, but then we must consider

$$g(\alpha) = \alpha \left[\sqrt{b - \alpha^2} - \sqrt{a - \alpha^2} \right] = \frac{d - c}{<} 0.$$

The function $g : [-\sqrt{a}, 0] \to \mathbb{R}$ is negative and increasing with $g(-\sqrt{a}) = -\sqrt{a}\sqrt{b - a}$ and $g(0) = 0$. The equation $g(\alpha) = (d - c)/2$ will have a unique negative solution α with $-\sqrt{a} < \alpha < 0$ if and only if

$$\frac{d - c}{2} > -\sqrt{a}\sqrt{b - a}.$$

That is, the condition

$$\frac{c - d}{2} < \sqrt{a}\sqrt{b - a}$$

must hold for there to exist a C^2 extremal when $c > d$.

Overall, we can summarize the condition for the existence of a unique C^2 extremal as

$$|d - c| < 2\sqrt{a}\sqrt{b - a}.$$

(c) This part is essentially already done above, but this is a good place to summarize the situation:

(0) If $c = d$, then the unique C^2 weak extremal is $u(x) \equiv c$.

(i) If $c < d$ and

$$d - c < 2\sqrt{a}\sqrt{b - a},$$

then the unique C^2 weak extremal is

$$u(x) = c + 2\alpha \left[\sqrt{x - \alpha^2} - \sqrt{a - \alpha^2} \right]$$

where α is the unique solution of the equation

$$\alpha \left[\sqrt{b - \alpha^2} - \sqrt{a - \alpha^2} \right] = \frac{d - c}{2}.$$

satisfying $0 < \alpha < \sqrt{a}$.

(ii) If $c > d$ and

$$c - d < 2\sqrt{a}\sqrt{b - a},$$

then the unique C^2 weak extremal is

$$u(x) = c + 2\alpha \left[\sqrt{x - \alpha^2} - \sqrt{a - \alpha^2} \right]$$

where α is the unique solution of the equation

$$\alpha \left[\sqrt{b - \alpha^2} - \sqrt{a - \alpha^2} \right] = \frac{d - c}{2}$$

satisfying $-\sqrt{a} < \alpha < 0$.

(d) I'm going to take $a = 1$, $b = 3$, $c = 1$ and $d = 2$. This falls into case (i) above since

$$\frac{d-c}{2} = \frac{1}{2} < \sqrt{2} = \sqrt{a}\sqrt{b-a}.$$

Plotting $f = f(\alpha)$ for $0 \leq \alpha \leq 1$ in this case, along with $(d-c)/2 = 1/2$, we find the unique root is around $\alpha = 0.6$; see Figure 1 below. Using Mathematica's `FindRoot`, we get an approximation $\alpha \doteq 0.60484$.

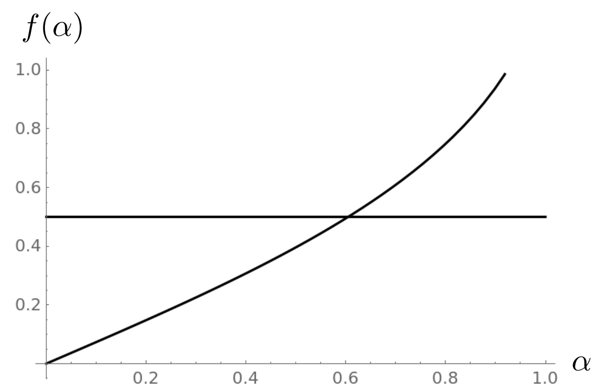

Figure 1: A plot of $f = f(\alpha)$.

In Figure 2, we have plotted the C^2 extremal (or at least our numerical approximation of it) on the left. On the right in Figure 2, we have plotted some quadratic competitors. These are given by

$$u(x) = \alpha x^2 + \beta x + \gamma$$

where we use $\alpha = -0.2, -0.1, 0, 0.1, 0.5, 1$ as a parameter with

$$\beta = \frac{d - c - (b^2 - a^2)\alpha}{b - a} \qquad \text{and} \qquad \gamma = d - \alpha b^2 - \beta b.$$

Note, this parameter α has nothing to do with the parameter α used in the discussion of the extremal.

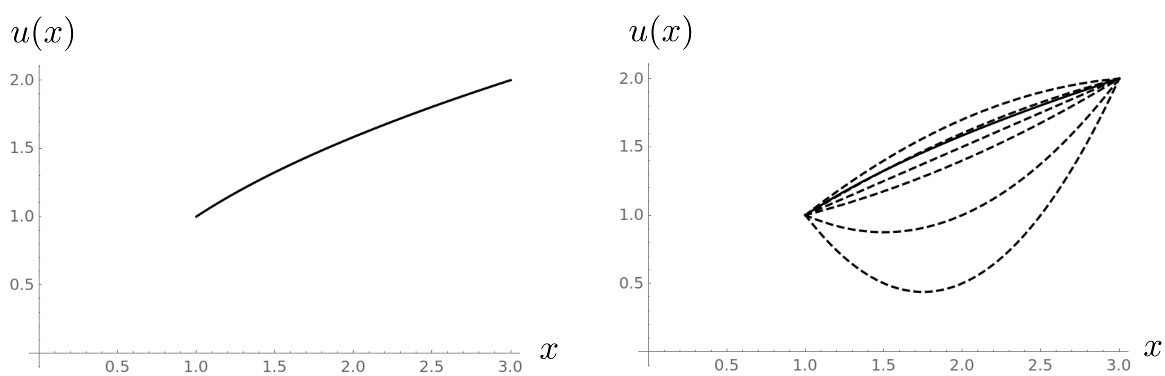

Figure 2: A plot of the C^2 extremal $u = u(x)$ on the left. The plot of u with some quadratic competitors on the right.

In Figure 3, we have plotted the functional values $\mathcal{F}[u]$ associated with each of the competitors and the value associated with the extremal (as a horizontal line of comparison).

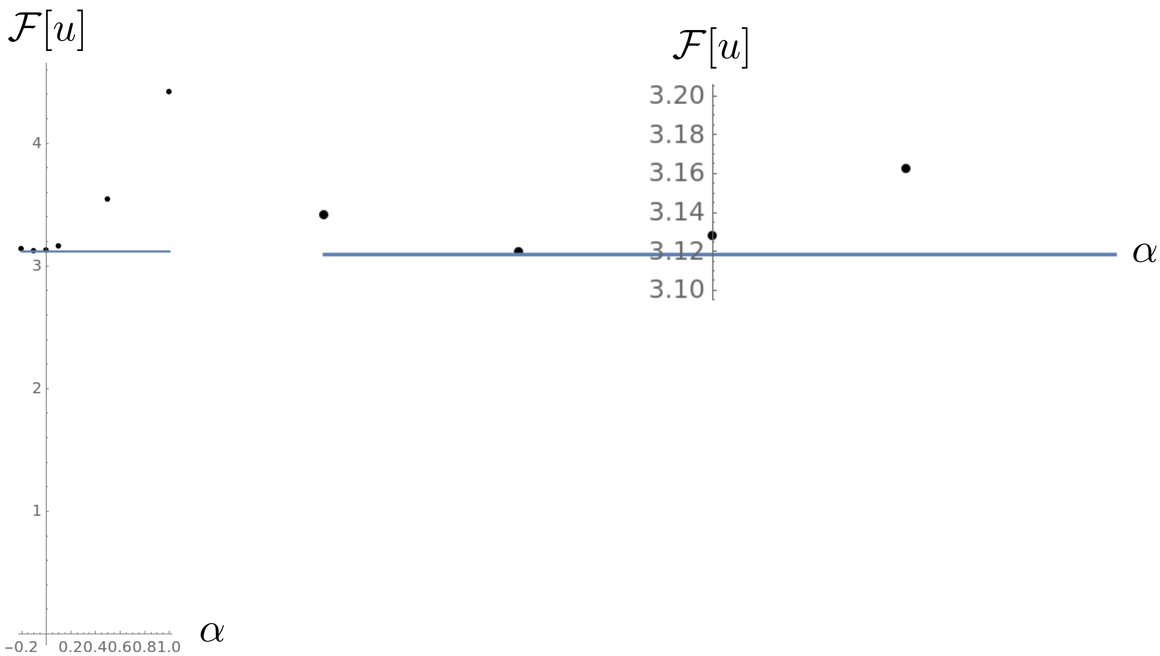

Figure 3: Functional values. All values are shown on the left. The lowest four competitors are shown on the right.

The C^2 extremal we have found numerically seems to be at least a local minimizer. In particular the quadratic competitors do not give lower values for $\mathcal{F}$, and the quadratic competitor u_1 corresponding to $\alpha = -0.1$, which is closest to the C^2 extremal, gives the closest functional value with

$$\mathcal{F}[u] \doteq 3.11635 < \mathcal{F}[u_1] \doteq 3.1195.$$

Note: I previously had posted a solution of this problem with an error in integration involving the introduction of a factor 1/2 instead of the correct factor of 2. Without knowing about the error, I mentioned the possibility of piecewise C^2 minimizers with corners, or so called "broken extremals." In the mean time I looked up the Erdmann corner conditions and realized that this was not what was happening. Fortunately, Ching-Lun Tai found my integration error, so I have written a corrected solution above.

For those who followed any of the discussion on piecewise smooth minimizers, I offer this nice variational problem from Hans Sagan:

The function $u : [-1, 1] \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$

is the unique piecewise C^1 minimizer of

$$\mathcal{F}[u] = \int_{-1}^{1} [u(x)]^2 [1 - u'(x)]^2 \, dx$$

in

$$\mathcal{A} = \{u \in \sqsubset^1[-1,1] : u(-1) = 0, \ u(1) = 1\}$$

where $\sqsubset^1[a,b]$ denotes the piecewise C^1 functions on the interval $[a,b]$, that is these functions satisfy $u \in C^0[a,b]$ (they are continuous) and there is some partition

$$a = x_0 < x_1 < \cdots < x_k = b$$

such that the restriction of u to $[x_{j-1}, x_j]$ is in $C^1[x_{j-1}, x_j]$ for $j = 1, 2, \ldots, k$. (These functions can have corners with different derivatives from the left and right at a point.) It's a little bit tricky to show the function above is the unique minimizer, but it is easy to see that it is a minimizer.

10. (Boas 9.2.3) Let a, b, c, and d be fixed **positive** real numbers with $0 < a < b$ and set

$$\mathcal{A} = \{u \in C^1[a,b] : u(a) = c, \ u(b) = d\}.$$

Consider the functional $\mathcal{E} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{E}[u] = \int_a^b x \sqrt{1 - u'(x)^2} \, dx.$$

(a) Compute the first variation $\delta\mathcal{E}_u : C_c^\infty(a,b) \to \mathbb{R}$.

(b) Determine the conditions under which there exist functions $u \in C^2(a,b) \cap \mathcal{A}$ for which $\delta\mathcal{E}_u \equiv 0$.

(c) Assuming the conditions you determined in part (b) for the existence of a C^2 weak extremal in $\mathcal{A}$, find all C^2 weak extremals.

(d) Take specific values for a, b, c, and d satisfying the conditions you found in part (b). Compare the functional values obtained for the C^2 weak extremals from part (c) (applied with these specific values for a, b, c, and d) to other values $\mathcal{E}[u]$ obtained from other admissible functions.

Solution:

(a) The first variation of this functional is given by

$$\delta\mathcal{E}_u[\phi] = -\int_a^b x \frac{u'}{\sqrt{1 - u'^2}} \phi' \, dx.$$

We should notice right away from this, that we need a restriction $|u'| < 1$ on our admissible functions. Furthermore, the mean value theorem tells us that this condition will be immediately violated for all $u \in \mathcal{A}$ unless

$$\frac{|d - c|}{b - a} < 1. \tag{4}$$

This is presumably the condition anticipated in part (b) below, though we have to see if this condition is sufficient to give us C^2 extremals. It is certainly necessary. Continuing with the assumption $u \in C^2[a,b]$, we can integrate by parts to obtain

$$\delta\mathcal{E}_u[\phi] = \int_a^b \left(x\frac{u'}{\sqrt{1-u'^2}} \right)' \phi\, dx \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

(b) Now we seek to show that if (4) holds, then we obtain a C^2 weak extremal, that is a solution of the ODE

$$\frac{u'}{\sqrt{1-u'^2}} = \frac{\alpha}{x}$$

for some constant α and some $u \in C^2[a,b]$ with $u(a) = c$ and $u(b) = d$.

Noting that the funtion α/x maintains a single sign which is the same as that of α, since $0 < a \le x \le b$, we see that any C^2 extremal must be monotone with $u' > 0$ is $\alpha > 0$ and $u' < 0$ if $\alpha < 0$. With this in mind, we can solve for u' to obtain:

$$u' = \frac{\alpha}{\sqrt{x^2+\alpha^2}} = \frac{\alpha}{|\alpha|}\frac{1}{\sqrt{(x/\alpha)^2+1}} = \frac{\alpha^2}{|\alpha|}\frac{d}{dx}\sinh^{-1}(x/\alpha).$$

It is perhaps easier to parse this expression if we consider cases. First of all, this integration is only valid if $\alpha \ne 0$. So we might consider

(0) $\alpha = 0$. In this case, our condition for a C^2 extremal tightens to $c = d$ since the only solutions are constant functions u for which $u' \equiv 0$. In this case, however, (when $c = d$) we clearly have a unique C^2 extremal $u \equiv c$. This extremal, however, is not a local minimizer. For example, if we take

$$u_\epsilon(x) = c + \epsilon(x-a)(x-b) = c + \epsilon[x^2 - (a+b)x + ab]$$

with $0 < |\epsilon| < 1/(b-a)$ then

$$\mathcal{E}[u] = \int_a^b x\,\sqrt{1-\epsilon^2(2x-a-b)^2}\, dx < \int_a^b x\, dx = \mathcal{E}[u_0].$$

(i) $\alpha > 0$ corresponding to $d > c$. In this case,

$$u(x) = c + \alpha\left[\sinh^{-1}\left(\frac{x}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right)\right].$$

A C^2 extremal must satisfy

$$f(\alpha) = \alpha\left[\sinh^{-1}\left(\frac{b}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right)\right] = d - c.$$

We claim that under our assumption $0 < d - c < b - a$, this equation has a unique positive solution α. For this, we need to establish some properties of $f = f(\alpha)$.

Lemma 1

$$\lim_{\alpha \searrow 0} f(\alpha) = 0.$$

Proof: Using the change of variables $\beta = 1/\alpha$ we can consider the limit

$$\lim_{\beta \nearrow \infty} \frac{\sinh^{-1}(b\beta) - \sinh^{-1}(a\beta)}{\beta}. \tag{5}$$

It's of course worth knowing that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

so that $\sinh^{-1}(\beta)$ is increasing and asymptotic to $\ln(2\beta)$ as $\beta \nearrow +\infty$. See Figure 4.

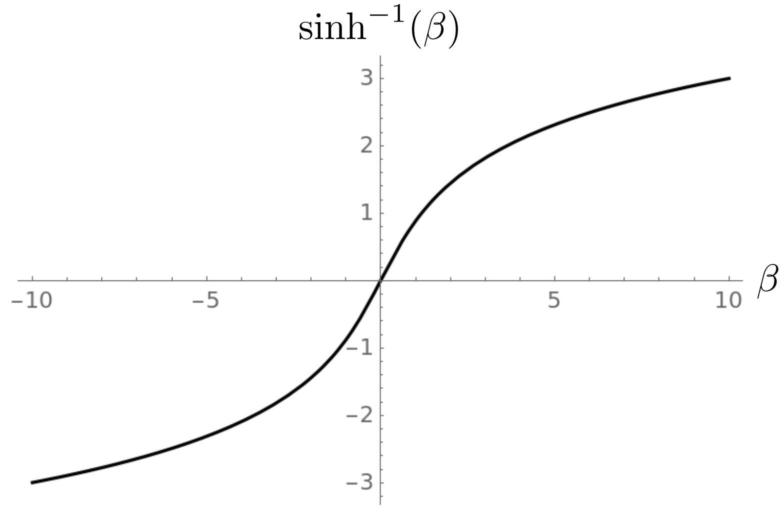

Figure 4: The graph of the inverse hyperbolic sine function.

This means, in particular, that the numerator in (5) is positive. We can also see that the numerator is increasing. In fact,

$$\begin{aligned} \frac{d}{d\beta}\left[\sinh^{-1}(b\beta) - \sinh^{-1}(a\beta)\right] &= \frac{b}{\sqrt{1+b^2\beta^2}} - \frac{a}{\sqrt{1+a^2\beta^2}} \\ &= \frac{1}{\sqrt{1/b^2+\beta^2}} - \frac{1}{\sqrt{1/a^2+\beta^2}} \\ &> 0 \end{aligned}$$

because $1/a^2 > 1/b^2$. Furthermore, the numerator in (5) is **bounded**, and this fact makes the assertion of (5) and our lemma obvious. However, it's not so obvious that this quantity is bounded. In fact, it has a rather interesting limiting value, so let's consider it carefully. First we'll take the hyperbolic sine of the quantity in question:

$$\sinh\left[\sinh^{-1}(b\beta) - \sinh^{-1}(a\beta)\right] = b\beta\sqrt{a^2\beta^2+1} - a\beta\sqrt{b^2\beta^2+1}.$$

Here we have used the difference formula for sinh and the fact that $\cosh(\sinh^{-1}(\beta)) = \sqrt{1+\beta^2}$ because $\cosh^2 x - \sinh^2 x = 1$. Continuing, we can write this quantity as

$$\begin{aligned}\beta\left[b\sqrt{a^2\beta^2+1} - a\sqrt{b^2\beta^2+1}\right] &= \frac{\beta(b^2-a^2)}{b\sqrt{a^2\beta^2+1}+a\sqrt{b^2\beta^2+1}} \\ &= \frac{b^2-a^2}{b\sqrt{a^2+1/\beta^2}+a\sqrt{b^2+1/\beta^2}}.\end{aligned}$$

Taking the limit in the last expression as $\beta \nearrow 0$, we find

$$\lim_{\beta\nearrow} \sinh\left[\sinh^{-1}(b\beta) - \sinh^{-1}(a\beta)\right] = \frac{b^2-a^2}{2ab}.$$

Consequently,

$$\lim_{\beta\nearrow}\left[\sinh^{-1}(b\beta) - \sinh^{-1}(a\beta)\right] = \sinh^{-1}\left(\frac{b^2-a^2}{2ab}\right) < \infty.$$

We have established the lemma. $\square$

Exercise 1 *We could have avoided getting the explicit limit of the numerator above. If the limit of the numerator is bounded (as we have shown), then clearly the assertion of (5) is correct. The alternative is that the numerator limits to* $+\infty$ *(which doesn't actually happen, but let's say we are going to ignore that for the moment and also avoid proving it). In this alternative case, we could apply L'Hopital's rule. Apply L'Hopital's rule under this assumption, and show you get zero for the limit.*

Note: One must be very careful with this sort of thing. If you assume the numerator tends to infinity, and it actually doesn't, then it is sometimes possible to apply L'Hopital's rule (incorrectly) and get the wrong limit. It just happens that you get zero in this misapplication of L'Hopital's rule.

Exercise 2 *Show* $f : (0,\infty) \to \mathbb{R}$ *by*

$$f(\alpha) = \alpha\left[\sinh^{-1}\left(\frac{b}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right)\right]$$

is differentiable (from the right) at $\alpha = 0$. *Find the value of* $f'(0^+)$.

Lemma 2

$$\lim_{\alpha\nearrow\infty} f(\alpha) = b - a.$$

Proof: Using the same change of variables $\beta = 1/\alpha$, the limit as $\beta \searrow 0$ of the expression in (5) is indeed an indeterminate form $0/0$, and so

$$\begin{aligned}\lim_{\beta\searrow 0}\frac{\sinh^{-1}(b\beta)-\sinh^{-1}(a\beta)}{\beta} &= \lim_{\beta\searrow 0}\left[\frac{b}{\sqrt{1+b^2\beta^2}} - \frac{a}{\sqrt{1+a^2\beta^2}}\right] \\ &= b - a. \qquad \square\end{aligned}$$

Lemma 3 *$f : [0, \infty) \to \mathbb{R}$ by*

$$f(\alpha) = \alpha \left[\sinh^{-1}\left(\frac{b}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right) \right]$$

is increasing.

Proof: I'm afraid this one is a little tricky, though maybe you can find an easier way to see it. First, notice that f is increasing if and only if

$$\frac{f(\alpha)}{a} = \frac{\alpha}{a} \left[\sinh^{-1}\left(\frac{b}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right) \right]$$

is increasing. (The left endpoint a is just a positive constant.) Now, f/a is increasing in α if and only if f/a is increasing as a function of $\beta = \alpha/a$ (chain rule), and writing f/a as a function of $\beta = \alpha/a$, we get a function $g : [0, \infty) \to \mathbb{R}$ by

$$g(\beta) = \beta \left[\sinh^{-1}\left(\frac{t}{\beta}\right) - \sinh^{-1}\left(\frac{1}{\beta}\right) \right]$$

where $t = b/a$ is a positive constant—which we are about to think of as a variable.

We wish to show $g'(\beta) > 0$, so we compute:

$$\begin{aligned} g'(\beta) &= \sinh^{-1}\left(\frac{t}{\beta}\right) - \sinh^{-1}\left(\frac{1}{\beta}\right) \\ &\quad + \beta \left[\frac{-t/\beta^2}{\sqrt{1 + t^2/\beta^2}} - \frac{-1/\beta^2}{\sqrt{1 + 1/\beta^2}} \right] \\ &= \sinh^{-1}\left(\frac{t}{\beta}\right) - \sinh^{-1}\left(\frac{1}{\beta}\right) \\ &\quad - \frac{t}{\sqrt{\beta^2 + t^2}} + \frac{1}{\sqrt{\beta^2 + 1}}. \end{aligned}$$

This may look pretty bad, but we're about to do something clever which will make things a bit simpler. Remember we want this quantity to be positive. Let's compute

$$\begin{aligned} \frac{\partial}{\partial t} g'(\beta) &= \frac{1/\beta}{\sqrt{1 + t^2/\beta^2}} - \left[\frac{1}{\sqrt{\beta^2 + t^2}} - \frac{t^2}{(\beta^2 + t^2)^{3/2}} \right] \\ &= \frac{t^2}{(\beta^2 + t^2)^{3/2}} \\ &> 0. \end{aligned}$$

Let's now think about what this means. We have a family of functions $g = g(\beta; t)$ depending smoothlly on a parameter $t = b/a > 1$. In fact, there is no singularity at $t = b/a = 1$, and in that case, we get $g(\beta; 1) \equiv 0$. That is, this limiting function is not increasing, but it is constant with $g'(\beta; 1) \equiv 0$. Now, when we increase t so that $t > 1$, the value of $g'(\beta; t)$ increases from zero. This means it becomes positive and $g' = g'(\beta) = g'(\beta; t) > 0$ for $t > 1$. $\square$

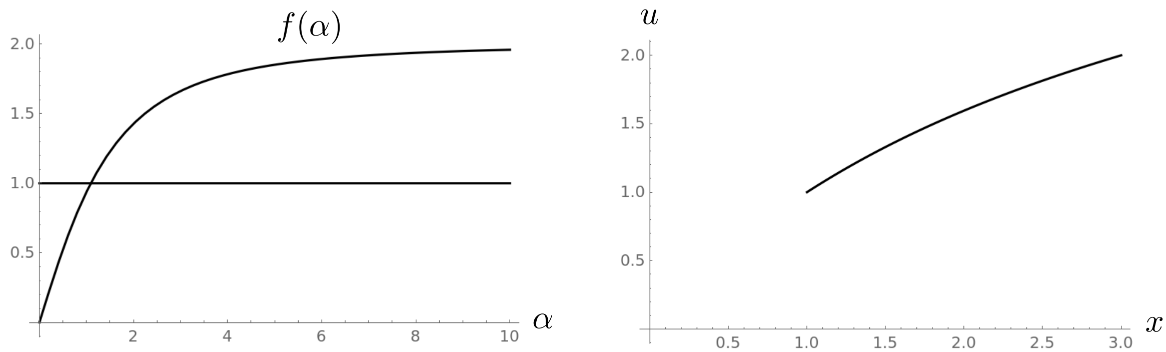

Figure 5: The graph of $f = f(\alpha)$ (left). A C^2 weak extremal (right). Both are for the parameters $a = 1$, $b = 3$, $c = 1$, $d = 2$.

We have established that the function $f = f(\alpha)$ has graph like that indicated on the left in Figure 5. f is increasing with $f(0) = 0$ and supremum $b - a$.

Exercise 3 *The method used above to prove Lemma 3 may seem unfamiliar, but I think you can also use it to prove* $f''(\alpha) < 0$*, which in view of the first two lemmas is an alternative approach to showing* $f'(\alpha) > 0$.

Using $a = 1$, $b = 3$, $c = 1$, and $d = 2$, we see, as indicated on the left in Figure 5, that the root of $f(\alpha) = (d - c) = 1$ is close to $\alpha = 1$. Numerically, we find $\alpha \doteq 1.09358$. The corresponding extremal is indicated on the right in Figure 5.

(ii) The function $f = f(\alpha)$ in this case is an even function, and I think you can check that if $c > d$ and $c - d < b - a$, then there is a unique negative α for which

$$f(\alpha) = c - d.$$

With this negative solution α, the unique C^2 extremal is given by the same formula:

$$u(x) = c + \alpha \left[\sinh^{-1}\left(\frac{x}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right) \right].$$

(c) Again, this is essentially done, so we just summarize:

(0) If $c = d$, then the unique C^2 weak extremal is $u(x) \equiv c$.

(i) If $0 < d - c < b - a$, then the unique weak extremal is

$$u(x) = c + \alpha \left[\sinh^{-1}\left(\frac{x}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right) \right]$$

where $\alpha > 0$ is the unique positive solution of

$$\alpha \left[\sinh^{-1}\left(\frac{b}{\alpha}\right) - \sinh^{-1}\left(\frac{a}{\alpha}\right) \right] = d - c.$$

(i) If $0 < c - d < b - a$, then the unique weak extremal is

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(d) None of the C^2 extremals above are minimizers of the functional $\mathcal{E}$.

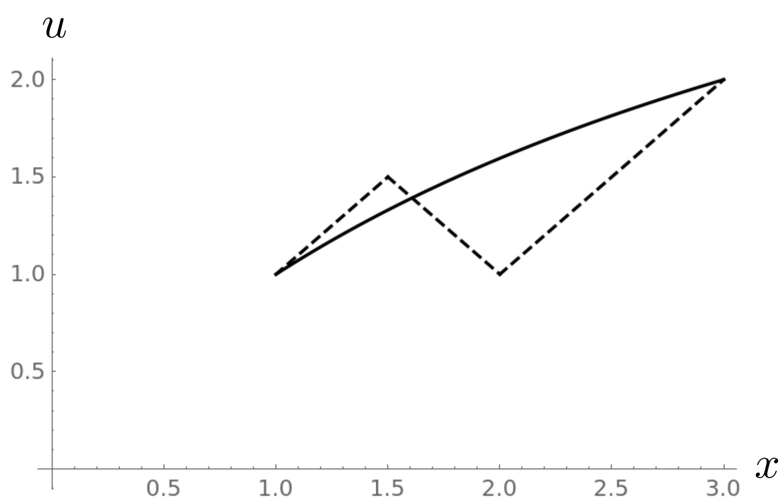

Figure 6: A minimizer for $a = 1$, $b = 3$, $c = 1$, $d = 2$ (dashed).

Any piecewise differentiable function with $|u'| \equiv 1$ where the derivative is defined minimizes $\mathcal{E}$ giving $\mathcal{E}[u] = 0$. One example,

$$u(x) = \begin{cases} x, & 1 \le x \le 1.5, \\ 3 - x, & 1.5 \le x \le 2 \\ x - 1, & 2 \le x \le 3 \end{cases}$$

is illustrated in Figure 6. There is obviously no uniqueness of minimizers in this case. The C^2 weak extremals for this functional are probably all local maximizers, if not global maximizers. This is clearly the case when $c = d$. Notice that the extremal we found for $a = 1$, $b = 3$, $c = 1$, and $d = 2$ is steeper when x is smaller and less steep when x is greater. This corresponds to maximizing the integrand factor $\sqrt{1 - u'^2}$ taking into account its weight of x.