Math 6702, Assignment 2 = Exam 1

Introduction

1. (Exercise 8 in the notes "Introduction" from 2020) What is the first order system equivalent to the ODE
$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x)?$$
Fully justify your answer.

2. (Exercise 24 in the notes "Introduction" from 2020) Find a system of first order equations equivalent to the hyperbolic PDE
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

§4.2 Power Series

3. (Boas 4.2.2,5) Find the power series expansions for
   (a) $\cos(x+y)$ and
   (b) $\sqrt{1+xy}$.

4. The **Taylor expansion** of a function $f \in C^\infty(\mathbb{R})$ at $x_0 \in \mathbb{R}$ is given by
$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j. \tag{1}$$
Here $f^{(j)}$ denotes the j-th (ordinary) derivative of f as usual:
$$f^{(j)} = \frac{d^j f}{dx^j}.$$
A function $f \in C^\infty(\mathbb{R})$ is said to be **real analytic** in the interval $I = (x_0 - r, x_0 + r)$ if the series in (1) converges for each $x \in I$ and
$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j.$$
The set of real analytic functions is denoted by C^ω. Verify that $\cos x$ is real analytic on $\mathbb{R}$, i.e., $\cos \in C^\omega(\mathbb{R})$.

5. Find a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^\infty(\mathbb{R}) \backslash C^\omega(\mathbb{R})$. Hint: Take $x_0 = 0$ and $f(x) \equiv 0$ for all $x \le 0$. Then (try to) define $f(x)$ for $x > 0$ so that all the derivatives $f^{(j)}(0)$ are zero, but the values of $f(x)$ for $x > 0$ are nonzero. This is a pretty hard problem if you've never seen such a function before.

6. The **Taylor expansion** of a function $u \in C^\infty(U)$ at $\mathbf{x}_0 \in U \subset \mathbb{R}^n$ is given by

$$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta. \tag{2}$$

There are a lot of things in this expansion formula which are probably new to you. Don't freak out. First, just compare (2) to (1) and observe that these two formulas are the "same" or at least sort of the same, so (on the face of it) this is a pretty cool formula, if it has some sensible meaning—and it does. The exercise will lead you through what it means.

(a) In this expansion formula $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a **multi-index**, which simply means

$$\beta \in \mathbb{N}^n = \{(m_1, \ldots, m_n) : m_1, \ldots, m_n \in \mathbb{N}\} \qquad \text{where} \qquad \mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

The derivative $D^\beta u$ denotes the partial derivative taken β_j times with respect to x_j for each $j = 1, 2, \ldots, n$:

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}.$$

The "length" of a multi-index β is defined by

$$|\beta| = \sum_{j=1}^{n} \beta_j.$$

Find all the multi-indices $\beta \in \mathbb{N}^3$ with $|\beta| = 2$.

(b) Write down all the second partials of a function $u : \mathbb{R}^3 \to \mathbb{R}$ in terms of multi-indices. Your answers should look like this:

$$D^{(2,0,0)} u = \frac{\partial^2 u}{\partial x^2}$$

and you should get five more for a total of six.

(c) Now let's back up a dimension to $\mathbb{R}^2$. The expansion for $f(x, y)$ given by Boas on page 192 has second order terms

$$\frac{1}{2!} \left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2 f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right].$$

The corresponding second order terms in (2) are

$$\sum_{|\beta|=2} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

where $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{x} = (x, y)$. To see that these are the same, you need to know the definition of the **factorial** of a multi-index, and you need to know how to take **multi-index powers** of a vector variable. Here are the definitions for $\beta \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$\beta! = \beta_1!\beta_2! \cdots \beta_n!.$$
$$\mathbf{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}.$$

Show that the second order terms given by Boas for a function of two variables are the same ones you get from the formula given in (2) when $n = 2$.

7. Given an open set $U \subset \mathbb{R}^n$, a function $u \in C^\infty(U)$ is said to be **real analytic** if for each $\mathbf{x}_0 \in U$, there exists some $r > 0$ such that the series in (2) converges for each $\mathbf{x} \in B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < r\}$ and

$$u(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta$$

for $\mathbf{x} \in B_r(\mathbf{x}_0) \cap U$. The set of **real analytic functions** on an open set $U \subset \mathbb{R}^n$ is denoted by $C^\omega(U)$. Find a function $u \in C^\infty(\mathbb{R}^n)\backslash C^\omega(\mathbb{R}^n)$.

Remark on notation: It is usual to denote the center of expansion of a power series in one variable by x_0 as in (1). For comparison of (2) to (1), we have used $\mathbf{x}_0$ as the (vector) center of expansion in the multivariable expansion. This causes a certain inconvenience when writing down the coordinates in higher dimensions. For $n = 2$ as in part (c) of problem 6, one can use $\mathbf{x}_0 = (x_0, y_0)$, and this approach can work for $n = 3$ as well with $\mathbf{x}_0 = (x_0, y_0, z_0)$. For general n, however, one usually resorts to something unpleasant like

$$\mathbf{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0).$$

To further understand the unpleasantness of this expression for the coordinates, you may write out the multi-index power $\mathbf{x}_0^\beta$. My preferred alternative is to replace $\mathbf{x}_0$ with $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, though some continuity of notation is lost between (1) and (2).

8. Repeat Boas' Problem 4.2.2 (given above as Problem 3) using the multi-index Taylor expansion formula.

Calculus of Variations

9. (Boas 9.2.1) Let a, b, c, and d be fixed **positive** real numbers with $0 < a < b$ and set

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = c,\ u(b) = d\}.$$

Consider the functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{F}[u] = \int_a^b \sqrt{x(1 + u'(x)^2}\, dx.$$

(a) Compute the first variation $\delta\mathcal{F}_u : C_c^\infty(a, b) \to \mathbb{R}$.

(b) Determine the conditions under which there exist functions $u \in C^2(a,b) \cap \mathcal{A}$ for which $\delta\mathcal{F}_u \equiv 0$. Hint: Consider possibilities for the ratio d/c along with the quantity $v(x) = \sin\psi = u'/\sqrt{1+u'^2}$ (the sine of the inclination angle).

(c) Assuming the conditions you determined in part (b) for the existence of a C^2 weak extremal in $\mathcal{A}$, find all C^2 weak extremals.

(d) Take specific values for a, b, c, and d satisfying the conditions you found in part (b). Compare the functional values obtained for the C^2 weak extremals from part (c) (applied with these specific values for a, b, c, and d) to other values $\mathcal{F}[u]$ obtained from other admissible functions.

Solution:

(a)

$$\delta\mathcal{F}_u[\phi] = \int_a^b \sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\phi'\,dx.$$

If $u \in C^2(a,b)$, then we can integrate by parts to write the first variation in the form

$$\delta\mathcal{F}_u[\phi] = \sqrt{b}\frac{u'(b)}{\sqrt{1+u'(b)^2}}\phi(b) - \sqrt{a}\frac{u'(a)}{\sqrt{1+u'(a)^2}}\phi' - \int_a^b \left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)\phi\,dx.$$

If we restrict to $\phi \in C_c^\infty(a,b)$, then the boundary terms vanish, and we have

$$\delta\mathcal{F}_u[\phi] = -\int_a^b \left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)'\phi\,dx.$$

(b) This is a little more complicated than the hint suggests. The Euler-Lagrange equation is

$$\left(\sqrt{x}\frac{u'}{\sqrt{1+u'^2}}\right)' = 0.$$

This means that if we have a solution then there must be a constant α such that

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}} = \frac{\alpha}{\sqrt{x}}. \tag{3}$$

The first observation is that the right side cannot change signs. This means any solution is either strictly increasing (if $\alpha > 0$) or strictly decreasing (if $\alpha < 0$). There is also the possibility that $\alpha = 0$. If $\alpha = 0$, then we must have $u' \equiv 0$ which means u is constant and to get a C^2 solution we must have $c = u(a) = u(b) = d$. So this is one condition where we do get a solution:

$$\text{If } c = d, \text{ then } u(x) \equiv c \text{ is a solution,}$$

and this is the unique C^2 extremal in this case.

We also know that if $c < d$ (i.e., $c/d < 1$, then the solution (if there is one) must be increasing and $\alpha > 0$. Similarly, if $d < c$, then the only possibility is $\alpha < 0$. Thus, we have two cases to consider.

Let's consider the case $c < d$. Then we can assume $\alpha > 0$.

The function

$$\sin\psi(x) = \frac{\alpha}{\sqrt{x}}$$

is well-defined, smooth, positive, and decreasing on $[a, b]$ with maximum value

$$0 < \frac{\alpha}{\sqrt{a}} = \sin\psi(a) < 1.$$

Therefore, we must have $0 < \alpha < \sqrt{a}$. Again, if we do have a solution, then we should be able to integrate (3) as follows: We first rearrange the equation as

$$u' = \frac{\alpha/\sqrt{x}}{\sqrt{1-\alpha^2/x}} = \frac{\alpha}{\sqrt{x-\alpha^2}} = \frac{\alpha}{2}\frac{d}{dx}\sqrt{x-\alpha^2}.$$

Then integration gives

$$u(x) = u(a) + \frac{\alpha}{2}\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right] = c + \frac{\alpha}{2}\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right].$$

In particular the second boundary condition requires $u(b) = d$ or

$$f(\alpha) = \alpha\left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right] = 2(d-c) > 0.$$

The function $f : [0, \sqrt{a}] \to \mathbb{R}$ is smooth with

$$f(0) = 0, \quad f(\sqrt{a}) = \sqrt{a}\sqrt{b-a} > 0,$$

and

$$f'(\alpha) = \left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right]\left(1 + \frac{\alpha^2}{\sqrt{(b-\alpha^2)(a-\alpha^2)}}\right) > 0.$$

In particular, f is increasing, and the equation $f(\alpha) = 2(d-c)$ will have a unique positive solution α with $0 < \alpha < \sqrt{a}$ if and only if

$$2(d-c) < f(\sqrt{a}) = \sqrt{a}\sqrt{b-a}.$$

Thus, the condition

$$2(d-c) < \sqrt{a}\sqrt{b-a}$$

is required for there to exist a C^2 extremal when $c < d$.

If $c > d$, then we must have $\alpha < 0$. In this case, $\sin\psi(x)$ is negative and increasing with minimum value $\sin\psi(a) = \alpha/\sqrt{a} < 0$. We must have $\alpha/\sqrt{a} \geq -1$, so

$$-\sqrt{a} < \alpha < 0.$$

The integration proceeds in the same way, but then we must consider

$$g(\alpha) = \alpha\left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right] = 2(d-c) < 0.$$

The function $g : [-\sqrt{a}, 0] \to \mathbb{R}$ is negative and increasing with $g(-\sqrt{a}) = -\sqrt{a}\sqrt{b-a}$ and $g(0) = 0$. The equation $g(\alpha) = 2(d-c)$ will have a unique negative solution α with $-\sqrt{a} < \alpha < 0$ if and only if

$$2(d-c) > -\sqrt{a}\sqrt{b-a}.$$

That is, the condition

$$2(c-d) < \sqrt{a}\sqrt{b-a}$$

must hold for there to exist a C^2 extremal when $c > d$.

Overall, we can summarize the condition for the existence of a unique C^2 extremal as

$$2|d-c| < \sqrt{a}\sqrt{b-a}.$$

(c) This part is essentially already done above, but this is a good place to summarize the situation:

(0) If $c = d$, then the unique C^2 weak extremal is $u(x) \equiv c$.

(i) If $c < d$ and

$$2(d-c) < \sqrt{a}\sqrt{b-a},$$

then the unique C^2 weak extremal is

$$u(x) = c + \frac{\alpha}{2}\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right]$$

where α is the unique solution of the equation

$$\alpha\left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right] = 2(d-c).$$

satisfying $0 < \alpha < \sqrt{a}$.

(ii) If $c > d$ and

$$2(c-d) < \sqrt{a}\sqrt{b-a},$$

then the unique C^2 weak extremal is

$$u(x) = c + \frac{\alpha}{2}\left[\sqrt{x-\alpha^2} - \sqrt{a-\alpha^2}\right]$$

where α is the unique solution of the equation

$$\alpha\left[\sqrt{b-\alpha^2} - \sqrt{a-\alpha^2}\right] = 2(d-c)$$

satisfying $-\sqrt{a} < \alpha < 0$.

(d) I'm going to take $a = 1$, $b = 3$, $c = 1$ and $d = 3/2$. This falls into case (i) above since

$$2(d-c) = 1 < \sqrt{2} = \sqrt{a}\sqrt{b-a}.$$

Plotting $f = f(\alpha)$ for $0 \le \alpha \le 1$ in this case, along with $2(d-c) = 1$, we find the unique root is around $\alpha = 0.9$; see Figure 1 below. Using Mathematica's `FindRoot`, we get an approximation $\alpha \doteq 0.92388$.

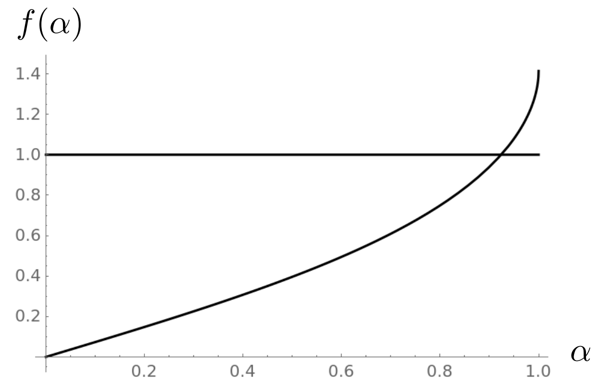

Figure 1: A plot of $f = f(\alpha)$.

In Figure 2, we have plotted the C^2 extremal (or at least our numerical approximation of it) on the left. On the right in Figure 2, we have plotted some quadratic competitors. These are given by

$$u(x) = \alpha x^2 + \beta x + \gamma$$

where we use $\alpha = -0.2, -0.1, 0, 0.1, 0.5, 1$ as a parameter with

$$\beta = \frac{d - c - (b^2 - a^2)\alpha}{b-a} \qquad \text{and} \qquad \gamma = d - \alpha b^2 - \beta b.$$

Note, this parameter α has nothing to do with the parameter α used in the discussion of the extremal.

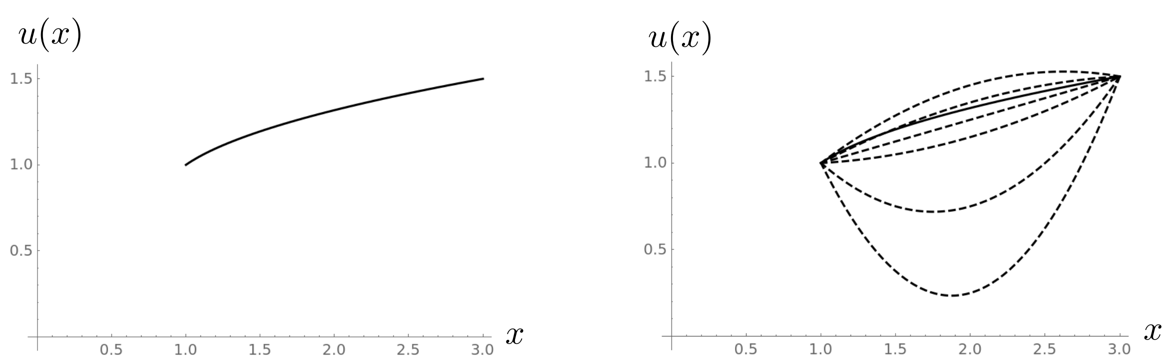

Figure 2: A plot of the C^2 extremal $u = u(x)$ on the left. The plot of u with some quadratic competitors on the right.

In Figure 3, we have plotted the functional values $\mathcal{F}[u]$ associated with each of the competitors and the value associated with the extremal (as a horizontal line of comparison).

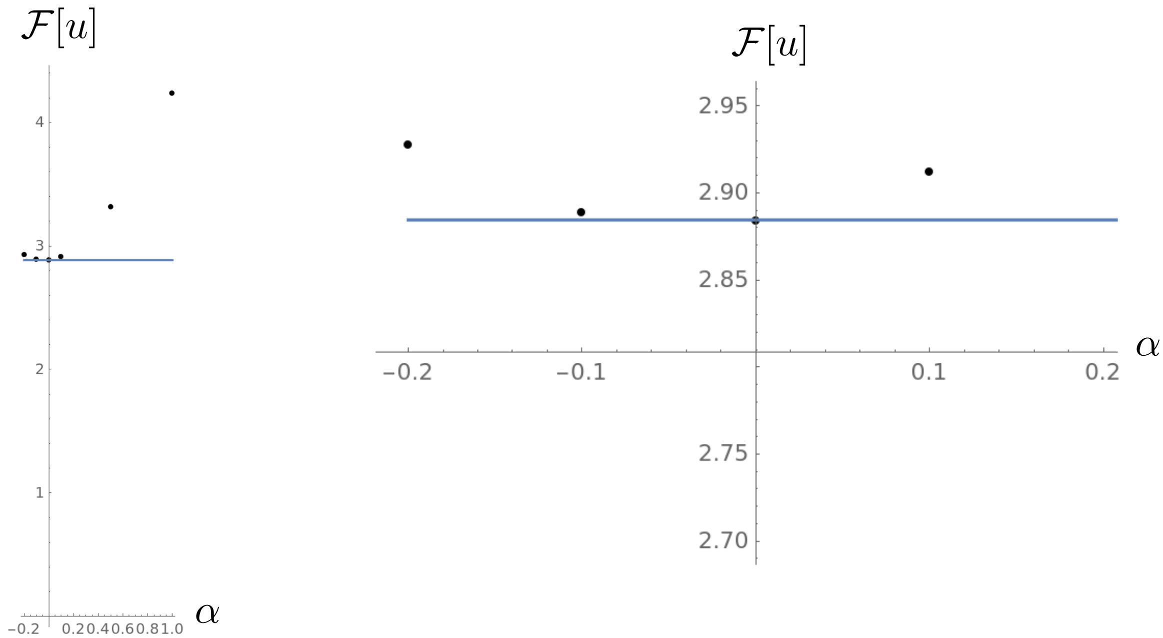

Figure 3: Functional values. All values are shown on the left. The lowest four competitors are shown on the right.

It is a little troubling that the straight line (corresponding to competitor parameter $\alpha = 0$) seems to have a functional value lower then our extremal. The appearance does not seem to be a misrepresentation. If u is the extremal, then we find

$$\mathcal{F}[u] \doteq 2.88443.$$

If

$$u_1(x) = \frac{1}{4}\,x + \gamma = \frac{1}{4}\,x + \frac{3}{4},$$

then

$$\mathcal{F}[u_1] \doteq 2.88353.$$

It is not entirely clear what to make of this calculation. The value we get with a straight line, while apparently smaller, is rather close to the extremal value. It may be that the extremal is actually a minimizer, and we are seeing a lower value due to numerical innaccuracy/round-off error. There are two obvious possible sources: (1) calculation of the root α for the extremal and (2) the numerical integration giving the values for both the extremal and the straight line solution. The integration for the straight line solution can be carried out explicitly, and the value you get seems to match the $\mathcal{F}[u_1] \doteq 2.88353$, so if there is innaccuracy in the numerical integration, it is probably primarily associated with the extremal.

One would generally expect that if u is not at least a local minimizer, then there should be a continuous deformation of u leading to arbitrarily low values of $\mathcal{F}$ (or at least decreasing values for $\mathcal{F}$ with competitors leading to a function with some singularity). If that is what actually happens here, then we are not finding such a sequence (at least not an obvious one) with our quadratic competitors.

Here is at least one way to (try to) check to see if the extremal is a minimizer: Consider a perturbation $\phi = u - u_1$ and compute $\mathcal{F}[u + t\phi]$. We have done this in Figure 4.

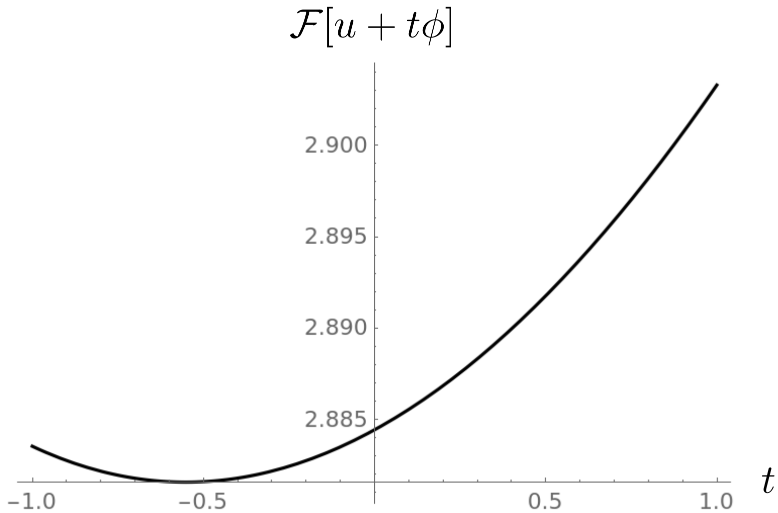

Figure 4: Functional values $\mathcal{F}[u+t\phi]$ with $t = 0$ corresponding to our (numerically computed) extremal and $t = -1$ corresponding to the straight line competitor. This is pretty strong evidence that we are not experiencing numerical approximation issues, and our extremal is not a minimizer. There should, however, be much lower functional values. It is not clear (to me) how to find them (at the moment).

Lessons from Problem 9: As I consider my work above on Problem 9, it strikes me that there are two likely possibilities:

1. Either I've made some fairly serious error in my calculations, and what I've got/concluded is fundamentally incorrect, or
2. The basic problem is the regularity assumption $u \in C^2[a, b]$ for minimizers, that is, the C^2 extremals are not minimizers because (even though you can find them) they have too much regularity.

There is also a third possibility that my work is basically correct, but there is some relatively minor error throwing off my ultimate conclusion. It doesn't feel like this to me. It does feel like one of the two possibilities I've listed and (at the moment—until I'm shown my error) especially the second one.

I will leave it to you to find error(s) in the solution above. Please find it/them. Let me make some comments based on the second possibility. What we have here is what appears to be a relatively minor modification of the length functional

$$\mathcal{L}[u] = \int_a^b \sqrt{1+u'^2}\,dx$$

for which we know everything works out quite nicely with unique minimizers in $C^2[a,b]$. We've just multiplied the Lagrangian by a positive C^∞ function $\sqrt{x}$. You may be inclined to point out that $\sqrt{x}$ has a singularity at $x = 0$, but note that we have specifically restricted attention to an interval $0 < a \le x \le b$, and we certainly have $h(x) = \sqrt{x}$ satisfying $h \in C^\infty[a,b] \cap C^\omega[a,b]$.

Nevertheless, if my work above is correct, we have a specific case ($a = 1$, $b = 3$, $c = 1$, $d = 3/2$) in which the unique extremal in $C^2[1,3]$ exists but is not a local minimizer. In such a case, a very likely possibility (which I do know happens in some other problems) is that there really is a minimizer, but it's not in $C^2[1,3]$. So, hopefully, you're inclined to ask:

1. Where is it (the minimizer)?
2. How do we find it?

Let me mention, by way of encouragement, that if what I'm about to suggest is correct, then finding the C^2 extremals (as I/we have done) is the right first step in finding the actual minimizers. So if we really want to find the minimizer for a problem like this, then we have wasted no effort. A good first lesson, however, is that calculus of variations problems (minimizing a function defined on an infinite dimensional set) is **much**, **much** more complicated than minimization over a finite dimensional set.

At this point it might be worth taking a step back and contemplating whether or not we really believe this problem should have a minimizer. One initial observation is that the functional is bounded below by

$$\mathcal{F}[u] = \int_a^b \sqrt{x}\sqrt{1+u'^2}\,dx \ge \sqrt{a}\mathcal{L}[u] \ge \sqrt{a}\sqrt{(b-a)^2+(d-c)^2}.$$

This is, of course, a good sign. And there are other results/theorems which one can consult to confirm the existence of a minimizer. I won't go into the details of those right now, but what I will say is that those theorems essentially never give the existence of a minimizer in the class $C^2[a,b]$. They always admit less regularity. What kind of regularity? Okay, I'll tell you.

Recall that our admissible class is/was

$$\mathcal{A} = \{u \in C^1[a,b] : u(a) = c,\ u(b) = d\}. \tag{4}$$

Now what I'm about to say is a little complicated and subtle, and it's going to sound contradictory, but I'm afraid it must be said: If a minimizer is in $C^1[a,b]$, then it is also in $C^2[a,b]$ or at least it still satisfies the Euler-Lagrange equation classically, and the Euler-Lagrange equation is a second order equation. What this tells you is that the minimizer is not (even) in $C^1[a, b]$. That sounds bad—very bad. After all, $C^1[a, b]$ might be considered the natural space on which the functional makes sense. Let's call the vector space $C^1[a, b]$ the **admissible background space**. It is the space of functions to which we add admissibility conditions; note carefully the role played by $C^1[a, b]$ in (4).

It turns out there is another nice admissible background space for which Lagrangian integral functionals make sense and **in which they often have minimizers**. Of course, this space is bigger than $C^1[a, b]$. It is the space of **piecewise** C^1 functions on $[a, b]$. I have a special notation I have invented for this admissible background space; you probably won't see it elsewhere, but I think it is a nice notation:

$$C^1[a, b] \subset \sqsubset^1[a, b].$$

A function u is in $\sqsubset^1[a, b]$ if $u \in C^0[a, b]$ and there exists a partition

$$a = x_0 < x_1 < \cdots < x_k = b$$

such that the restrictions

$$u\Big|_{x_{j-1} \le x \le x_j}$$

satisfy

$$u\Big|_{x_{j-1} \le x \le x_j} \in C^1[x_{j-1}, x_j] \qquad \text{for } j = 1, 2, \ldots, k.$$

A function in $\sqsubset^1[a, b]$ can have **corners** at the points of the partition. And sometimes minimizers have to have those. There is a nice theorem called the **Erdmann corner condition theorem** which tells you, in terms of your functional, when an extremal does not have, or cannot have, corners. I haven't checked the corner conditions for this functional to see if corners are ruled out, but I'm guessing they are not. This is what it feels like.

There is something else one should notice about the C^2 extremals of this problem (if I've got that part correct) which may be important and suggests a yet different, bigger, and more complicated admissible background space: The borderline extremals

$$u(x) = c + \frac{\alpha}{2}\left[\sqrt{x - \alpha^2} - \sqrt{a - \alpha^2}\right]$$

corresponding to $\alpha = \sqrt{a}$ are not in $C^1[a, b]$, and they are not in $\sqsubset^1[a, b]$ either. They are in the set of continuous functions admitting a partition as above such that the restrictions satisfy

$$u\Big|_{x_{j-1} \le x \le x_j} \in C^1(x_{j-1}, x_j) \qquad \text{for } j = 1, 2, \ldots, k.$$

This collection of functions is not good enough to have the functional be finite valued. But the particular extremals we have in there are functions for which the functional is well-defined, and we might need them. I'm going to go ahead and describe the appropriate admissible background space, though it's going to get a bit technical. Hopefully, the problem above illustrates why such contortions may be required when minimizing in infinite dimensions.

Let us say $u \in \sqsubset^1_{loc}[a,b]$ if $u \in C^0[a,b]$ and there exists a partition

$$a = x_0 < x_1 < \cdots < x_k = b$$

such that if U is an open interval compactly contained in (x_{j-1}, x_j) for some $j = 1, 2, \ldots, k$, then the restriction

$$u\big|_U \in C^1(U).$$

In words, $\sqsubset^1_{loc}[a,b]$ is the collection of continuous functions which are piecewise locally C^1, i.e., piecewise locally continuously differentiable.

As mentioned above, $\mathcal{F}[u]$ may not be finite valued for a function $u \in \sqsubset^1_{loc}[a,b]$. Therefore, we need to take a somewhat smaller set. One fairly common approach is to take as admissible background space

$$\mathcal{B} = \{u \in \sqsubset^1_{loc}[a,b] : \mathcal{F}[u] < \infty\}.$$

One small difficulty is that one needs to show this is a vector space, but I'm pretty sure that can be shown. An alternative (requiring a bit more mathematical background concerning integration and measurability) is to consider the vector space of **weakly differentiable** functions $H^1(a,b)$. These are measurable functions u with a weak derivative u' satisfying

$$\int_{(a,b)} |u'| < \infty.$$

While I haven't described these functions in full detail, it can be checked that when a function in $\sqsubset^1_{loc}[a,b]$ has a weak derivative, that derivative must agree with the (piecewise defined) classical derivative. Furthermore, I believe our admissible background space $\mathcal{B}$ satisfies

$$\mathcal{B} = \sqsubset^1_{loc}[a,b] \cap H^1(a,b).$$

At least this space, being the intersection of vector spaces, is definitely a vector space. Finally, then, we can restate our problem from the beginning:

> Minimize $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ by
>
> $$\mathcal{F}[u] = \int_{(a,b)} \sqrt{x}\sqrt{1+u'^2}$$
>
> where
>
> $$\mathcal{A} = \{u \in \sqsubset^1_{loc}[a,b] \cap H^1(a,b) : u(a) = c,\ u(b) = d\}.$$

Practically speaking, one would (at least start) by considering a single division point (x_1, y_1) with $a < x_1 < b$ and probably $c < y_1 < d$, though the latter restriction is probably not necessary to state here; I think it comes up as a necessary condition later.

The point is that, very often, **when corner points are required to find a minimizer, only one corner point is necessary**.

Also, it's worth recalling my comment above that a C^1 minimizer is also C^2. This assertion holds **piecewise**. That is to say, if you have a minimizer and the restriction of that minimizer to any open interval happens to be C^1, then it will also be C^2 there. In particular, we are going to assume there is one division point x_1 where a corner can occur. We will also assume the minimizer is C^1, and hence C^2, on the open intervals (a, x_1) and (x_1, b). This essentially allows us to use the work given in my solution above to minimize on each interval separately. The implication is: While some (and probably most) minimizers require corner points and/or gradient blow-ups, there are some special boundary values for which you don't need lower regularity (i.e., corners and/or gradient blow-ups). We already know one such case: $c = d$.

So, we take some $x_1 \in (a, b)$ and find the extremal for

$$\mathcal{F}_a[u] = \int_a^{x_1} \sqrt{x}\sqrt{1 + u'^2}\, dx$$

in

$$\mathcal{A}_a = \{u \in C^0[a, x_1] \cap C^2(a, x_1) \cap H^1(a, x_1) : u(a) = c,\ u(x_1) = y_1\}$$

using the approach given as my solution above. We also find the extremal for

$$\mathcal{F}_b[u] = \int_{x_1}^{b} \sqrt{x}\sqrt{1 + u'^2}\, dx$$

in

$$\mathcal{A}_b = \{u \in C^0[x_1, b] \cap C^2(x_1, b) \cap H^1(x_1, b) : u(x_1) = y_1,\ u(b) = d\}.$$

Then we piece these two extremals together to obtain a piecewise extremal $u = u(x; x_1, y_1)$. Finally, we consider

$$\phi(x_1, y_1) = \mathcal{F}[u(x; x_1, y_1)]$$

as a function of two real variables and minimize over an appropriate open set of points (x_1, y_1) in $(a, b) \times \mathbb{R} \subset \mathbb{R}^2$. The function you will get, which probably has an interior corner and maybe one or more integrable gradient blow-ups, is probably the minimizer for the problem.

I'm sure it sounds like a lot of work...but this is how it goes (or at least roughly how it can go) with minimization over infinite dimensional spaces.

10. (Boas 9.2.3) Let a, b, c, and d be fixed **positive** real numbers with $0 < a < b$ and set

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = c,\ u(b) = d\}.$$

Consider the functional $\mathcal{E} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{E}[u] = \int_a^b x\ \sqrt{1 - u'(x)^2)}\, dx.$$

(a) Compute the first variation $\delta\mathcal{E}_u : C_c^\infty(a,b) \to \mathbb{R}$.

(b) Determine the conditions under which there exist functions $u \in C^2(a,b) \cap \mathcal{A}$ for which $\delta\mathcal{E}_u \equiv 0$.

(c) Assuming the conditions you determined in part (b) for the existence of a C^2 weak extremal in $\mathcal{A}$, find all C^2 weak extremals.

(d) Take specific values for a, b, c, and d satisfying the conditions you found in part (b). Compare the functional values obtained for the C^2 weak extremals from part (c) (applied with these specific values for a, b, c, and d) to other values $\mathcal{E}[u]$ obtained from other admissible functions.