Annali di Matematica (2013) 192:329–348
DOI 10.1007/s10231-011-0226-x

Local quasi-concavity of the solutions of the heat equation with a nonnegative potential

Daniele Andreucci · Kazuhiro Ishige

Received: 17 May 2011 / Accepted: 11 October 2011 / Published online: 22 October 2011

Abstract In this paper we consider the Cauchy problem for the heat equation with a nonnegative potential decaying quadratically at the space infinity and investigate local concavity properties of the solution. In particular, we give a sufficient condition for the solution to be quasi-concave in a ball for any sufficiently large t, and discuss the optimality of the sufficient condition, identifying a threshold for the occurrence of local quasi-concavity.

Keywords Concavity of solutions of parabolic equations · Local quasi-concavity for large times · Heat equation with potential · Hot spots

Mathematics Subject Classification (2000) 35B40 · 35K05 · 35K15

1 Introduction

Consider the Cauchy problem for the heat equation with a potential,

$$\begin{cases} \partial_t u - \Delta u + V(|x|)u = 0 \text{ in} & \mathbf{R}^N \times (0, \infty), \\ u(x, 0) = \varphi(x) \qquad\qquad \text{in} & \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $N \geq 2$, $\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\mathrm{d}x)$, and V is a smooth nonnegative function behaving like

$$V(r) = \omega r^{-2}(1 + o(1)) \qquad \text{as} \qquad r \to \infty \tag{1.2}$$

D. Andreucci (✉)
Dipartimento di Scienze di Base e Applicate per l'Ingegneria,
University of Rome La Sapienza, via A. Scarpa 16, 00161 Rome, Italy
e-mail: andreucci@dmmm.uniroma1.it

K. Ishige
Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

for some $\omega \geq 0$. More precisely, we assume the following condition on the potential V: there exists a nonnegative constant ω such that

$$\begin{cases} \text{(i)} \quad V = V(|x|) \in C^1\left(\mathbf{R}^N\right), \\ \text{(ii)} \;\; V(r) \geq 0 \quad \text{in} \quad [0, \infty), \\ \text{(iii)} \; \limsup_{r\to\infty} r^\kappa \left|V(r) - \omega r^{-2}\right| < \infty \quad \text{for some } \kappa > 2, \\ \text{(iv)} \; \limsup_{r\to\infty} r^3 |V'(r)| < \infty. \end{cases} \tag{V}$$

Then, Cauchy problem (1.1) has a unique solution u in the function space

$$X := L^\infty\left(0, \infty : L^2\left(\mathbf{R}^N\right)\right) \cap L^2\left(0, \infty : H^1\left(\mathbf{R}^N\right)\right)$$

(see for example [14] and [15]). In this paper, under condition (V), we consider the large time behavior of the solution u and investigate local (spatial) concavity properties of the solution u.

The study of concavity properties of solutions of parabolic equations is a classical subject and has been studied extensively by many mathematicians (see for example [1–9], [16–26], and references therein). Among others, in [6], Brascamp and Lieb proved that, for any nonnegative solution u of the Cauchy problem for the heat equation,

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad \mathbf{R}^N \times (0, \infty), \qquad u(x, 0) = \varphi(x) \geq 0 \quad \text{in} \quad \mathbf{R}^N, \tag{1.3}$$

$u(\cdot, t)$ is log-concave in $\mathbf{R}^N$ for all $t > 0$ if the initial function φ is log-concave in $\mathbf{R}^N$. This preservation of log-concavity holds for the Cauchy–Dirichet problem of nonlinear parabolic equations of several types in strictly convex bounded domains (see for example [9] and [22]). Furthermore, if φ is supported in the ball of radius $R > 0$, then the solution $u(\cdot, t)$ of (1.3) is log-concave for all $t \geq R^2/2$ without the log-concavity of the initial function φ (see [24]). In addition, it is known that the quasi-concavity, that is, the convexity of superlevel sets of the solution, is not preserved by heat flow if $N \geq 2$ (see [16] and [17]). For the heat equation with a potential,

$$\partial_t u - \Delta u + V(x)u = 0 \qquad \text{in} \qquad \mathbf{R}^N \times (0, \infty), \tag{1.4}$$

in [2–4], Borell assumed that $V^{-1/2}$ is concave in $\mathbf{R}^N$, and studied the concavity properties of the fundamental solutions and the ground states of (1.4). However, as far as we know, there are no results treating concavity properties of the solutions of the heat equation with a potential satisfying (1.2).

On the other hand, in [10–13], the second author of this paper and Kabeya considered Cauchy problem (1.1) under condition (V), and studied the large time behavior of the solutions and their hot spots. The large time behavior of the solutions and their hot spots heavily depend on the behavior of potential V at the space infinity, and they are characterized by the harmonic functions for the operator $-\Delta + V$.

In this paper we develop the arguments in [10–13], and study local concavity properties of the solution $u(\cdot, t)$ of (1.1). In particular, for any $R > 0$, we give a sufficient condition for the solution $u(\cdot, t)$ of (1.1) to be quasi-concave in the ball $B(0, R) := \{x \in \mathbf{R}^N : |x| < R\}$ for all sufficiently large t (see Theorem 1 (c)). Furthermore, we discuss the optimality of the sufficient condition and prove the existence of the threshold number ω_* discriminating between the cases where the solution has a local quasi-concavity property for all sufficiently large t or not (see Theorem 2 and (1.6)).

Before stating the main results of this paper, we give the notion of γ-concavity ($\gamma \in [-\infty, \infty]$), log-concavity, and quasi-concavity for nonnegative functions in a convex domain. See also [6] and [21].

Definition 1 Let $\gamma \in [-\infty, \infty]$ and Ω be a convex domain in $\mathbf{R}^N$. We say that a nonnegative function f in Ω is γ-concave in Ω if the support of f in Ω is convex and f satisfies

(i) f is a constant function on the support of f for the case $\gamma = \infty$;
(ii) f^γ is concave on its support for the case $\gamma > 0$;
(iii) $\log f$ is concave on its support for the case $\gamma = 0$;
(iv) f^γ is convex on its support for the case $\gamma < 0$;
(v) all of the superlevel sets $\{x \in \Omega : f(x) > \lambda\}$ with $\lambda > 0$ are convex for the case $\gamma = -\infty$.

More precisely, we say that a nonnegative function f in Ω is γ-concave in Ω if f satisfies the inequality

$$f((1-\lambda)x + \lambda y) \geq M_\gamma(f(x), f(y), \lambda)$$

for all $x, y \in \Omega$ and $\lambda \in (0, 1)$, where

$$M_\gamma(a_0, a_1, \lambda) = \begin{cases} \left[(1-\lambda)a_0^\gamma + \lambda a_1^\gamma\right]^{1/\gamma} & \text{for } \gamma \neq -\infty, 0, \infty, \\ a_0^{1-\lambda} a_1^\lambda & \text{for } \gamma = 0, \\ \max\{a_0, a_1\} & \text{for } \gamma = \infty, \\ \min\{a_0, a_1\} & \text{for } \gamma = -\infty, \end{cases}$$

if $a_0, a_1 > 0$ and $M_\gamma(a_0, a_1, \lambda) = 0$ if $a_0 = 0$ or $a_1 = 0$. Furthermore, we say that f is log-concave in Ω if f is 0-concave in Ω and that f is quasi-concave in Ω if f is $-\infty$-concave in Ω.

We remark that if $-\infty \leq \gamma' \leq \gamma \leq \infty$ and f is γ-concave in Ω, then f is γ'-concave in Ω and that quasi-concavity is the weakest notion in the concavity properties given in Definition 1. For further properties of γ-concavity, see for example [21, Section 2].

We introduce some notation. For any $\omega \geq 0$, let $\alpha(\omega)$ be the nonnegative root of the algebraic equation $\alpha(\alpha + N - 2) = \omega$, that is,

$$\alpha(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2}. \tag{1.5}$$

Let $\{\omega_k\}_{k=0}^\infty$ be the eigenvalues of the Laplace–Beltrami operator $-\Delta_{\mathbf{S}^{N-1}}$ on the $(N-1)$ dimensional sphere $\mathbf{S}^{N-1}$ such that $0 = \omega_0 < \omega_1 < \cdots$, that is, $\omega_k = k(N - 2 + k)$, $k = 0, 1, 2, \ldots$. Here we remark that

$$\alpha(\omega_k) = k, \qquad k = 0, 1, 2, \ldots.$$

Then, by (1.5) we can define a positive constant ω_* uniquely by

$$\alpha(\omega_* + \omega_2) = \alpha(\omega_*) + 1 \tag{1.6}$$

(see Lemma 2.1). For any $k = 0, 1, 2, \ldots$, let l_k and $\{Q_{k,i}\}_{i=1}^{l_k}$ be the dimension and the complete orthonormal system of the eigenspace corresponding to ω_k, respectively. Then, we see $l_0 = 1$ and $l_1 = N$ and have

$$Q_{0,1}\left(\frac{x}{|x|}\right) = q_0, \qquad Q_{1,i}\left(\frac{x}{|x|}\right) = q_1 \frac{x_i}{|x|} \quad (i = 1, \ldots, N), \tag{1.7}$$

where

$$q_0 = \left(\int_{\mathbf{S}^{N-1}} \mathrm{d}\sigma \right)^{-1/2}, \qquad q_1 = q_0\sqrt{N}, \tag{1.8}$$

and $\mathrm{d}\sigma$ is the surface measure on $\mathbf{S}^{N-1}$. On the other hand, under condition (V), there exists a solution U of the ordinary differential equation

$$U'' + \frac{N-1}{r}U' - \left(V(r) + \frac{\omega_k}{r^2} \right) U = 0 \quad \text{in} \quad (0,\infty) \quad \text{such that} \quad \limsup_{r\to 0} U(r) < \infty \tag{1.9}$$

(see Sect. 2.1). Then, there exists a constant d such that

$$U(r) = d\eta(r)(1+o(1)) \quad \text{as} \quad r \to \infty,$$

where

$$\eta(r) = \begin{cases} \log(2+r) & \text{if} \quad (N,\omega,k) = (2,0,0), \quad V \not\equiv 0, \\ r^{\alpha(\omega+\omega_k)} & \text{otherwise} \end{cases} \tag{1.10}$$

(see Sect. 2.1). We denote by U_k the solution of (1.9) satisfying $\lim_{r\to\infty} U_k(r)/\eta(r) = 1$. Furthermore, for any $k = 0, 1, 2, \dots$ and $i = 1, \dots, l_k$, we put

$$\mathcal{U}_{k,i}(x) = U_k(|x|) Q_{k,i}\left(\frac{x}{|x|} \right). \tag{1.11}$$

Then, $\mathcal{U}_{k,i}(x)$ is a stationary solution of (1.1), that is, $\mathcal{U}_{k,i}$ satisfies

$$-\Delta \mathcal{U}_{k,i} + V(|x|)\mathcal{U}_{k,i} = 0 \quad \text{in} \quad \mathbf{R}^N, \tag{1.12}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}^N} u(x,t)\mathcal{U}_{k,i}(x)\mathrm{d}x = 0, \qquad t > 0,$$

for the solution u of (1.1). These functions $\mathcal{U}_{k,i}(x)$, in particular, $\mathcal{U}_{0,1}(x)$ and $\mathcal{U}_{1,i}(x)$ ($i = 1, \dots, N$), play an important role in the study of the large time behavior of the solutions of (1.1) and their hot spots (see [10–12]).

Now we are ready to state our main results of this paper. We first give a sufficient condition and a necessary condition for the solution $u(\cdot,t)$ to have a local concavity property.

Theorem 1 *Assume condition* (V) *and let $u \in X$ be a solution of problem* (1.1) *such that*

$$\varphi \in L^2(\mathbf{R}^N, e^{|x|^2/4}\mathrm{d}x), \qquad M := \int_{\mathbf{R}^N} \varphi(x)\mathcal{U}_{0,1}(x)\mathrm{d}x > 0. \tag{1.13}$$

Then, for any $R > 0$, there exists a constant T with the following properties:

(a) *$u(x,t) > 0$ for all $x \in B(0,R)$ and $t \geq T$;*
(b) *if the function $u(\cdot,t)$ is quasi-concave in $B(0,R)$ for some $t \geq T$, then $V(r) \equiv 0$ on $[0,R]$;*
(c) *if $V(r) \equiv 0$ in $[0,R]$ and $\omega < \omega_*$, then the function $u(\cdot,t)$ is concave in $B(0,R)$ for all $t \geq T$.*

Theorem 1 implies that the solution u of (1.1) is not necessarily quasi-concave in a ball $B(0, R)$ for any sufficiently large t even if the initial function φ is log-concave in $\mathbf{R}^N$ and that the flatness of the potential around the origin induces the local quasi-concavity of the solution u for all sufficiently large t. Furthermore, we see that, under condition (V), if u is quasi-concave in $\mathbf{R}^N$ for all sufficiently large t, then $V(r) \equiv 0$ on $[0, \infty)$, that is, u is a solution of the heat equation.

Next we study the local concavity property of the solution of (1.1) for the case $\omega \geq \omega_*$. By Theorems 1 (c) and 2 we see that, if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, then the case $\omega = \omega_*$ is a threshold discriminating between the cases where the solution $u(\cdot, t)$ is quasi-concave in $B(0, R)$ for all sufficiently large t. We prove that, for the case $\omega \geq \omega_*$, even if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, $u(\cdot, t)$ is not necessarily quasi-concave in $B(0, R)$ for any sufficiently large t.

Theorem 2 *Assume condition* (V) *and that* $V(r) \equiv 0$ *on* $[0, R]$ *for some* $R > 0$. *Then, there holds the following:*

(a) *if either* $\omega > \omega_*$ *or* $(N, \omega) = (2, \omega_*)$, *there exists a nonnegative solution* $u \in X$ *with the initial function* $\varphi \in C_0\left(\mathbf{R}^N\right)$ *such that* $u(\cdot, t)$ *is not quasi-concave in* $B(0, R)$ *for any sufficiently large* t;

(b) *if* $\omega = \omega_*$ *and* $N \geq 3$, *there exists a solution* $u \in X$ *with the initial function* $\varphi \in C_0\left(\mathbf{R}^N\right)$ *satisfying*

$$\int_{\mathbf{R}^N} \varphi(x)\,\mathcal{U}_{0,1}(x)\mathrm{d}x > 0$$

such that $u(\cdot, t)$ *is not quasi-concave in* $B(0, R)$ *for any sufficiently large* t.

We give the idea of the proofs of our theorems. In [10, Theorem 1.1], [11, Theorem 1.1], and [12, Theorem 1.1], the first asymptotic expansion of the solution of (1.1) has been already obtained, and there hold

$$\begin{cases} \lim\limits_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega)}u(x,t) = C_1 M U_0(|x|) & \text{if} \quad (N,\omega) \neq (2,0), \\ \lim\limits_{t\to\infty} t(\log t)^2 u(x,t) = C_2 M U_0(|x|) & \text{if} \quad (N,\omega) = (2,0), \end{cases} \tag{1.14}$$

uniformly for all x in any compact set of $\mathbf{R}^N$, where C_1 and C_2 are positive constants. However, geometric properties of the solution are not necessarily determined by the first asymptotic expansion of the solution. Indeed, if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, the function $U_0(|x|)$ is a constant function in $B(0, R)$, and the asymptotics (1.14) cannot determine the geometric properties of the solutions of (1.1). Therefore, if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, we have to study in detail the large time behavior of solution of (1.1). For the case $(N, \omega) \neq (2, 0)$, we follow the arguments in [10] and [11], and study the large time behavior of radial solutions $\{v_{k,i}\}$ of the equation

$$\partial_t v = \Delta v - \left(V(|x|) + \frac{\omega_k}{|x|^2}\right) v$$

(see also (2.1)). Then, we obtain

$$u(x,t) = c_0 M t^{-\frac{N}{2}-\alpha(\omega)} - c_0' t^{-\frac{N}{2}-\alpha(\omega)-1}|x|^2 + \sum_{i=1}^{N} c_i t^{-\frac{N}{2}-\alpha(\omega+\omega_1)} x_i + O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_2)}\right) \tag{1.15}$$

for all $x \in B(0, R)$ and all sufficiently large t, by using the fact: for any $k \in \mathbf{N}_0$,

$$U_k(r) = d_k r^k, \qquad r \in [0, R],$$

for some positive constant d_k if $V \equiv 0$ in $[0, R]$. Here c_0, c_0', c_1, ..., c_N are constants, in particular, due to $M > 0$, c_0, and c_0' are positive constants. Then, the second term in the right-hand side of the asymptotic expansion (1.15) yields the local concavity property of the solution u, and we have assertion (c) of Theorem 1 for the case $(N, \omega) \neq (2, 0)$. Similarly, for the case $(N, \omega) = (2, 0)$, we can obtain assertion (c) of Theorem 1 with the aid of the results of [12]. Thus, we have Theorem 1. On the other hand, for the case $\omega \geq \omega_*$, we have $\alpha(\omega + \omega_2) \leq \alpha(\omega) + 1$ (see Lemma 1), and we can choose an initial function for which the last term $O(t^{-N/2-\alpha(\omega+\omega_2)})$ in the right-hand side of (1.15) determines the geometric properties of the solution in $B(0, R)$. Then, we can prove Theorem 2.

The rest of this paper is organized as follows: In Sect. 2, we recall some preliminary results, which are obtained in [10–12]. In Sect. 3, we study the large time behavior of the radial solutions $v_{k,i}$ and their derivatives. Section 4 is devoted to the proof of Theorems 1 and 2.

2 Preliminaries

In this section we recall some results given in [10–12] and study the radial solution v_k of the problem

$$\begin{cases} \partial_t v = \Delta v - \left(V(|x|) + \dfrac{\omega_k}{|x|^2} \right) v & \text{in} \quad \mathbf{R}^N \times (0, \infty), \\ v(x, 0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \\ \limsup\limits_{x \to 0} |v(x, t)| < \infty \quad \text{for any} \quad t > 0, & \end{cases} \tag{2.1}$$

where $k = 0, 1, 2, \ldots$ and ϕ is a radial function in $\mathbf{R}^N$ such that $\phi \in L^2(\mathbf{R}^N, \rho \mathrm{d}y)$ with $\rho(y) = e^{|y|^2/4}$.

2.1 Solutions U_k of problem (1.9)

We consider problem (1.9) under condition (V). Since the function r^k is a solution of

$$U'' + \frac{N-1}{r} U' - \frac{\omega_k}{r^2} U = 0 \quad \text{in} \quad (0, \infty),$$

we apply the method of variation of constants and see that the solution U of problem (1.9) satisfies

$$U(r) = \mathrm{d}r^k + r^k \int_0^r s^{1-2k-N} \left(\int_0^s \tau^{N+k-1} V(\tau) U(\tau) \mathrm{d}\tau \right) \mathrm{d}s, \quad r \in (0, \infty), \tag{2.2}$$

for some constant d. This representation with the standard arguments in the ordinary differential equations implies the existence of the solution U of (1.9), which is uniquely determined by the limit of $U(r)/\eta(r)$ as $r \to \infty$ (see [10–12]). Here, $\eta = \eta(r)$ is the function given in (1.10). We denote by U_k the solution U of (1.9) satisfying

$$\lim_{r \to \infty} U(r)/\eta(r) = 1. \tag{2.3}$$

Then, (2.2) yields

$$U_k(r) = d_k r^k + r^k \int_0^r s^{1-2k-N} \left(\int_0^s \tau^{N+k-1} V(\tau) U_k(\tau) \mathrm{d}\tau \right) \mathrm{d}s, \quad r \in (0, \infty), \tag{2.4}$$

for some constant $d_k > 0$, and we see that

$$U_k(r) \geq d_k r^k \geq 0, \qquad U_k'(r) \geq k d_k r^{k-1} \geq 0 \tag{2.5}$$

for all $r \geq 0$. In particular, for the case $k = 0$, we have

$$U_0(r) \geq U_0(0) = d_0 > 0 \quad \text{for all} \quad r \geq 0. \tag{2.6}$$

Furthermore, since $\limsup_{r \to 0} U_k(r) < \infty$, for any $L > 0$, we have $\sup_{0<r<L} U_k(r) < \infty$, and by (2.4) we obtain

$$\left| \frac{\mathrm{d}^l}{\mathrm{d}r^l} U_k(r) \right| \leq C_k r^{k-l} \qquad (l = 0, 1, 2) \tag{2.7}$$

for all $r \in (0, L)$, where C_k is a constant depending on L and k. In addition, if $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, by (2.4) we have

$$U_k(r) = d_k r^k, \qquad r \in [0, R]. \tag{2.8}$$

For the details in the argument above, see [10, Lemma 2.2], [11, Lemma 2.1], and [12, Lemma 3.1]. Furthermore, we have:

Proposition 1 *Assume* (V) *and let* $k = 0, 1, 2, \ldots$. *Let* f *be a continuous function on* $[0, \infty)$ *and put*

$$F_k[f](r) = U_k(r) \int_0^r s^{1-N} [U_k(s)]^{-2} \left(\int_0^s \tau^{N-1} U_k(\tau) f(\tau) \mathrm{d}\tau \right) \mathrm{d}s. \tag{2.9}$$

Then, there holds the following:

(i) *for any solution* $v = v(r)$ *of*

$$U'' + \frac{N-1}{r} U' - \left(V(r) + \frac{\omega_k}{r^2} \right) U = f \quad \text{in} \quad (0, \infty), \tag{2.10}$$

satisfying $\limsup_{r \to 0} |v(r)| < \infty$, *there exists a constant* c *such that*

$$v(r) = c U_k(r) + F_k[f](r), \qquad r \in [0, \infty);$$

(ii) *if there exist a positive constant* R *and a monotone increasing function* $A = A(r)$ *on* $[0, R]$ *such that*

$$|f(r)| \leq A(r) U_k(r), \qquad r \in [0, R],$$

then

$$|F_k[f](r)| \leq \frac{A(r)}{2N} r^2 U_k(r), \quad \left| F_k[f]'(r) \right| \leq \frac{A(r)}{2N} [2r U_k(r) + r^2 U_k'(r)],$$

for all $r \in [0, R]$.

Proof Since function $F_k[f](r)$ is a solution of (2.10) such that $F_k[f](0) = 0$, we have assertion (i) of Proposition 1 by the uniqueness of the solution of problem (1.9). Assertion (ii) of Proposition 1 follows from the definition of $F_k[f]$ and the monotonicity of $U_k(r)$ (see (2.5)). □

Next we give a lemma on the definition of ω_*.

Lemma 1 *There exists a positive constant ω_* such that*

$$\alpha(\omega+\omega_2)-\alpha(\omega)\begin{cases} >1 & \text{if } \ 0\le\omega<\omega_*,\\ =1 & \text{if } \ \omega=\omega_*,\\ <1 & \text{if } \ \omega>\omega_*.\end{cases} \tag{2.11}$$

Proof Put $f(s) := \alpha(s+\omega_2)-\alpha(s)$ for $s \ge 0$. Then, $f(0) = \alpha(\omega_2)-\alpha(0) = 2 > 1$. By (1.5), we have

$$f(s) = \frac{1}{2}\left(\sqrt{(N-2)^2+4(s+\omega_2)}-\sqrt{(N-2)^2+4s}\right),$$

which implies $\lim_{s\to\infty} f(s) = 0$ and that $f'(s) < 0$ for all $s \ge 0$. Then, Lemma 1 follows. □

2.2 Solutions of problem (2.1)

Let $k = 0, 1, 2\dots$ and ϕ be a radial function in $L^2(\mathbf{R}^N, \rho dx)$. Then, under condition (V), there exists a unique classical and radial solution $v := S_k(t)\phi \in X$ of (2.1) such that

(i) $v(x,t)Q_{k,i}(x/|x|)$ is a solution of

$$\partial_t u - \Delta u + V(|x|)u = 0 \quad \mathbf{R}^N \times (0,\infty),$$

(ii) for any $1 \le q \le p \le \infty$ and $l = 1, 2, \dots$, there exists a constant C, independent of k, such that

$$\|v(t)\|_{L^p(\mathbf{R}^N)} \le Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\phi\|_{L^q(\mathbf{R}^N)}, \quad \|(\partial_t^l v)(t)\|_{L^2(\mathbf{R}^N)} \le Ct^{-l}\|\phi\|_{L^2(\mathbf{R}^N)},$$

for all $t > 0$.

Furthermore, we have the following proposition (see [10, Lemma 3.2], [11, Lemma 2.1], and [12, Lemma 3.3]).

Proposition 2 *Let $k = 0, 1, 2, \dots$ and ϕ be a radial function in $L^2(\mathbf{R}^N, \rho dx)$. Let $v = S_k(t)\phi \in X$ be a solution of (2.1) under condition (V). Then, there holds the following:*

(i) *let $(N, \omega, k) \ne (2, 0, 0)$ and assume that*

$$\|v(t)\|_{L^2(\mathbf{R}^N)} \le C_1 t^{-d}, \quad t > 0,$$

for some constants $C_1 > 0$ and $d \ge 0$. Then, for any $T > 0$ and any sufficiently small $\epsilon > 0$, there exists a constant C_2 such that

$$\left|\left(\partial_t^j v\right)(x,t)\right| \le C_1C_2 t^{-d-\frac{N}{4}-\frac{\alpha(\omega+\omega_k)}{2}-j}U_k(|x|)$$

for all $(x,t) \in D_\epsilon(T)$, where $j = 0, 1, 2$ and

$$D_\epsilon(T) := \{(x,t) \in \mathbf{R}^N \times [T,\infty) \,:\, |x| \le \epsilon(1+t)^{1/2}\};$$

(ii) *let* $(N, \omega, k) = (2, 0, 0)$ *and assume that*

$$\|v(t)\|_2 \le C_3(1+t)^{-\frac{1}{2}}[\log(2+t)]^{-d'}, \quad t > 0,$$

for some constants $C_3 > 0$ *and* $d' \ge 0$*. Then, for any* $T > 0$ *and any sufficiently small* $\epsilon > 0$*, there exists a constant* C_4 *such that*

$$\left|\left(\partial_t^j v\right)(x,t)\right| \le C_3 C_4 t^{-1-j}[\log(2+t)]^{-d'-1} U_0(|x|)$$

for all $(x, t) \in D_\epsilon(T)$*, where* $j = 0, 1, 2$.

Next we put

$$w(y,s) = (1+t)^{\frac{N}{2}} v(x,t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t). \tag{2.12}$$

Then, the function w satisfies

$$\partial_s w = Lw := L^* w - \tilde{V}(y,s) w \quad \text{in} \quad \mathbf{R}^N \times (0,\infty), \quad w(y,0) = \phi(y) \quad \text{in} \quad \mathbf{R}^N,$$

where

$$L^* w := \frac{1}{\rho}\operatorname{div}(\rho \nabla w) + \frac{N}{2} w - \frac{\omega + \omega_k}{|y|^2} w, \quad \tilde{V}(y,s) := e^s V(e^{\frac{s}{2}} y) - \frac{\omega}{|y|^2}, \quad \rho(y) := e^{\frac{|y|^2}{4}}.$$

The operator L^* has the following property (see [10, Lemma 3.3]).

Proposition 3 *Let* $\omega \ge 0$ *and* $k = 0, 1, 2, \ldots$*. Let* $\{\lambda_{k,i}\}_{i=0}^\infty$ *be the eigenvalues of*

$$\begin{cases} L^*\varphi = -\lambda\varphi & \text{in} \quad \mathbf{R}^N, \\ \varphi \text{ is a radial function in } \mathbf{R}^N \text{ with respect to } 0, \\ \varphi \in H^1(\mathbf{R}^N, \rho \mathrm{d}y), \end{cases} \tag{E}$$

such that $\lambda_{k,0} < \lambda_{k,1} < \cdots$*. Then,* $\lambda_{k,i} = (\alpha(\omega + \omega_k)/2) + i$ *and all the eigenvalues are simple. Furthermore, the eigenfunction* φ_k *corresponding to* $\lambda_{k,0}$ *is given by*

$$\varphi_k(y) = c_k |y|^{\alpha(\omega+\omega_k)} \exp\left(-\frac{|y|^2}{4}\right),$$

where c_k *is a positive constant such that* $\|\varphi_k\|_{L^2(\mathbf{R}^N, \rho \mathrm{d}y)} = 1$.

Then, we can expand the function $w(s)$ to the Fourier series of the eigenfunctions of problem (E) and have the following proposition on the large time behavior of the function w. In what follows, we write $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^N, \rho \mathrm{d}y)}$ for simplicity.

Proposition 4 *Assume the same conditions as in Proposition 2. Let* w *be a function defined by* (2.12)*. Then, there holds the following:*

(i) *let* $(N, \omega, k) \ne (2, 0, 0)$*. Then, there exists a positive constant* C_1 *such that*

$$\|w(s)\| \le C_1 e^{-\frac{\alpha(\omega+\omega_k)}{2}s} \|\phi\|, \quad s > 0. \tag{2.13}$$

Furthermore, for any $L > 1$*, there holds*

$$\lim_{s\to\infty} \left\| e^{\frac{\alpha(\omega+\omega_k)}{2}s} w(s) - a_k \varphi_k \right\|_{C^2(\{L^{-1} \le |y| \le L\})} = 0, \tag{2.14}$$

where

$$a_k := c_k \int_{\mathbf{R}^N} U_k(|x|)\phi(x)\mathrm{d}x. \tag{2.15}$$

Here c_k is the constant given in Proposition 3;

(ii) *let $(N,\omega,k) = (2,0,0)$. Then, there exists a positive constant C_2 such that*

$$\|w(s)\| \le C_2 s^{-1}\|\phi\|, \quad s > 0. \tag{2.16}$$

Furthermore, for any $L > 1$, there holds

$$\lim_{s\to\infty} \|sw(s) - 2a_0\varphi_0\|_{C^2(\{L^{-1}\le|y|\le L\})} = 0. \tag{2.17}$$

Assertion (i) of Proposition 4 is given by Proposition 3.2 in [10] and assertion (ii) by Lemma 3.6 and (3.33) in [12].

3 Behavior of the solution of (2.1)

In this section we study the large time behavior of the solution of (2.1) by using the results given in Sect. 2. We first give an upper estimate of the solution of (2.1).

Lemma 2 *Let $k = 0, 1, 2, \dots$ and ϕ be a radial function in $L^2(\mathbf{R}^N, \rho\mathrm{d}x)$. Then, for any $T > 0$ and any sufficiently small $\epsilon > 0$, there exists a constant C such that*

$$|[S_l(t)\phi](x)| \le \begin{cases} Ct^{-\frac{N}{2}-\alpha(\omega+\omega_k)}U_k(|x|)\|\phi\| & \textit{if} \quad (N,\omega,k) \ne (2,0,0), \\ Ct^{-\frac{N}{2}}(\log(t+2))^{-2}U_0(|x|)\|\phi\| & \textit{if} \quad (N,\omega,k) = (2,0,0), \end{cases} \tag{3.1}$$

for all $(x,t) \in D_\epsilon(T)$ and $l \ge k$.

Proof Let $l \ge k$. By the comparison principle, we have

$$|[S_l(t)\phi](x)| \le [S_l(t)|\phi|](x) \le [S_k(t)|\phi|](x) \tag{3.2}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. On the other hand, (2.13) and (2.16) imply that

$$\|S_k(t)|\phi|\|_2 \le \begin{cases} C_1(1+t)^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_k)}{2}}\|\phi\| & \text{if} \quad (N,\omega,k) \ne (2,0,0), \\ C_1(1+t)^{-\frac{N}{4}}[\log(t+2)]^{-1}\|\phi\| & \text{if} \quad (N,\omega,k) = (2,0,0), \end{cases}$$

for all $t > 0$, where C_1 is a constant. Then, by Proposition 2 and (3.2) we have inequality (3.1), and Lemma 2 follows. □

Next, combining Propositions 1 and 4 with Lemma 2, we have:

Lemma 3 *Let $k = 0, 1, 2, \dots$ and ϕ be a radial function in $L^2(\mathbf{R}^N, \rho\mathrm{d}x)$. Let $v = S_k(t)\phi$ be a solution of (2.1) under condition (V). Then, there exists a function $\zeta = \zeta(t)$ defined in $(0,\infty)$, satisfying*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta(t) = \begin{cases} (a_k c_k + o(1))\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}t^{-\frac{N}{2}-\alpha(\omega+\omega_k)} & \textit{if} \quad (N,\omega,k) \ne (2,0,0), \\ (4a_0c_0 + o(1))\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-1}(\log t)^{-2}] & \textit{if} \quad (N,\omega,k) = (2,0,0), \end{cases} \tag{3.3}$$

as $t \to \infty$ *for* $n = 0, 1$, *such that*

$$v(x,t) = \zeta(t)U_k(|x|) + \zeta'(t)F_k[U_k](|x|) + F_k\left[F_k[(\partial_t^2 v)(\cdot,t)]\right](|x|) \tag{3.4}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. *Furthermore, for any* $L > 0$, *there exist constants* C *and* T *such that*

$$\left|\partial_r^l F_k\left[F_k[(\partial_t^2 v)(\cdot,t)]\right](|x|)\right| \le \begin{cases} Ct^{-\frac{N}{2}-\alpha(\omega+\omega_k)-2}|x|^{k+4-l}\|\phi\| \\ \quad \textit{if} \quad (N,\omega,k) \neq (2,0,0), \\ Ct^{-3}[\log(2+t)]^{-2}|x|^{4-l}\|\phi\| \\ \quad \textit{if} \quad (N,\omega,k) = (2,0,0), \end{cases} \tag{3.5}$$

for all $(x,t) \in B(0,L) \times (T,\infty)$ *and* $l = 0, 1, 2$.

Proof By (2.1) and Proposition 1 (i) we can find a function $\zeta = \zeta(t)$ defined in $(0,\infty)$, satisfying

$$v(x,t) = \zeta(t)U_k(|x|) + F_k[(\partial_t v)(\cdot,t)](|x|), \quad (x,t) \in \mathbf{R}^N \times (0,\infty). \tag{3.6}$$

The smoothness of v and $F_k[\partial_t v]$ together with (3.6) implies $\zeta \in C^1(0,\infty)$ and

$$\partial_t v(x,t) = \zeta'(t)U_k(|x|) + F_k[(\partial_t^2 v)(\cdot,t)](|x|) \tag{3.7}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. Then, by (3.6) and (3.7) we have (3.4).

On the other hand, by Proposition 4 we have

$$\|v(t)\|_{L^2(\mathbf{R}^N)} \le \begin{cases} C_1 t^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_k)}{2}}\|\phi\| & \text{if} \quad (N,\omega,k) \neq (2,0,0), \\ C_1 t^{-\frac{1}{2}}[\log(2+t)]^{-1}\|\phi\| & \text{if} \quad (N,\omega,k) = (2,0,0), \end{cases} \tag{3.8}$$

for all $t > 0$, where C_1 is a constant. Let $T > 0$ and ϵ be a sufficiently small positive constant. Then, by Proposition 2 and (3.8) we see that there exists a constant C_2 such that

$$|(\partial_t^2 v)(x,t)| \le \begin{cases} C_2 t^{-\frac{N}{2}-\alpha(\omega+\omega_k)-2}U_k(|x|)\|\phi\| & \text{if} \quad (N,\omega,k) \neq (2,0,0), \\ C_2 t^{-3}[\log(2+t)]^{-2}U_0(|x|)\|\phi\| & \text{if} \quad (N,\omega,k) = (2,0,0), \end{cases}$$

for all $(x,t) \in D_\epsilon(T)$. This inequality together with Proposition 1 (ii) yields

$$\left|F_k[(\partial_t^2 v)(\cdot,t)](|x|)\right| \le \begin{cases} C_3 t^{-\frac{N}{2}-\alpha(\omega+\omega_k)-2}|x|^2 U_k(|x|)\|\phi\| \\ \quad \text{if} \quad (N,\omega,k) \neq (2,0,0), \\ C_3 t^{-3}[\log(2+t)]^{-2}|x|^2 U_0(|x|)\|\phi\| \\ \quad \text{if} \quad (N,\omega,k) = (2,0,0), \end{cases} \tag{3.9}$$

for all $(x,t) \in D_\epsilon(T)$, where C_3 is a constant. Then, by (2.7), (2.9), and (3.9) we obtain (3.5). Furthermore, by Lemma 2, (3.6), and (3.9) we have

$$\lim_{t\to\infty} \zeta(t) = 0. \tag{3.10}$$

It remains to prove (3.3). We first consider the case $(N,\omega,k) \neq (2,0,0)$. Then, by (2.3) we have

$$\lim_{r\to\infty} r^{-\alpha(\omega+\omega_k)}U_k(r) = 1,$$

and by (3.7) and (3.9), for any $\nu \in (0, \epsilon)$, we obtain

$$\begin{aligned} & t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}(\partial_t v)(x,t)\Big|_{|x|=\nu t^{1/2}} \\ & = t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}\left[\zeta'(t) + O(\nu^2 t^{-\frac{N}{2}-\alpha(\omega+\omega_k)-1})\right] U_k(\nu t^{1/2}) \\ & = \nu^{\alpha(\omega+\omega_k)} t^{\frac{N}{2}+\alpha(\omega+\omega_k)+1}\zeta'(t) + O(\nu^{\alpha(\omega+\omega_k)+2}) \end{aligned} \tag{3.11}$$

for all sufficiently large t. On the other hand, by Proposition 4 (i) and (2.12) we have

$$\begin{aligned} & \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_k)+l}{2}}(\nabla_x^l v)(x,t)\Big|_{|x|=\nu t^{1/2}} \\ & = \lim_{s\to\infty} e^{\frac{\alpha(\omega+\omega_k)}{2}s}(\nabla_y^l w)(y,s)\Big|_{|y|=\nu} = a_k(\nabla_y^l \varphi_k)(y)\Big|_{|y|=\nu}, \quad l = 0, 1, 2. \end{aligned} \tag{3.12}$$

Furthermore, by condition (V) (iii) and (3.12) with $l = 0$ we have

$$\begin{aligned} & \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}\left(V(|x|) - \frac{\omega}{|x|^2}\right) v(x,t)\Big|_{|x|=\nu t^{1/2}} \\ & = \lim_{t\to\infty} o\left(t^{\frac{N+\alpha(\omega+\omega_k)}{2}}\right) v(x,t)\Big|_{|x|=\nu t^{1/2}} = 0. \end{aligned} \tag{3.13}$$

Then, by (2.1), (3.12), and (3.13) we have

$$\begin{aligned} & \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}(\partial_t v)(x,t)\Big|_{|x|=\nu t^{1/2}} \\ & = \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}\left[\Delta v - \left(V(|x|) + \frac{\omega_k}{|x|^2}\right) v\right]\Big|_{|x|=\nu t^{1/2}} \\ & = \lim_{t\to\infty} t^{\frac{N+\alpha(\omega+\omega_k)}{2}+1}\left[\Delta v - \frac{\omega+\omega_k}{|x|^2} v\right]\Big|_{|x|=\nu t^{1/2}} = a_k\left[\Delta\varphi_k - \frac{\omega+\omega_k}{|y|^2}\varphi_k\right]_{|y|=\nu}. \end{aligned} \tag{3.14}$$

Since $\varphi_k(y) = c_k|y|^{\alpha(\omega+\omega_k)}e^{-|y|^2/4}$, we have

$$\left[\Delta\varphi_k - \frac{\omega+\omega_k}{|y|^2}\varphi_k\right]_{|y|=\nu} = -c_k\left[\frac{N}{2} + \alpha(\omega+\omega_k)\right]\nu^{\alpha(\omega+\omega_k)}(1 + O(\nu)).$$

Therefore, since ν is arbitrary, by (3.11) and (3.14) we have

$$\lim_{t\to\infty} t^{\frac{N}{2}+\alpha(\omega+\omega_k)+1}\zeta'(t) = -a_k c_k\left[\frac{N}{2} + \alpha(\omega+\omega_k)\right], \tag{3.15}$$

and obtain (3.3) for the case $l = 1$. Furthermore, since

$$\zeta(t) = -\int_t^\infty \zeta'(s)\mathrm{d}s$$

by (3.10), (3.15) implies (3.3) for the case $l = 0$. Thus, the proof of (3.3) for the case $(N, \omega, k) \neq (2, 0, 0)$ is complete.

Next we consider the case $(N, \omega, k) = (2, 0, 0)$. Similarly to (3.11), for any sufficiently small $\nu > 0$, we have

$$\begin{aligned} t^2(\log t)(\partial_t v)(x,t)\Big|_{|x|=\nu t^{1/2}} &= t^2(\log t)\left[\zeta'(t) + O\left(\nu^2 t^{-2}[\log(2+t)]^{-2}\right)\right] U_0(\nu t^{1/2}) \\ &= \frac{1}{2}t^2(\log t)^2\zeta'(t) + O(\nu^2) \end{aligned} \tag{3.16}$$

as $t \to \infty$. Furthermore, similarly to (3.14), by Proposition 4 (ii) we have

$$\begin{aligned} &\lim_{t\to\infty} t^2(\log t)(\partial_t v)(x,t)\Big|_{|x|=\nu t^{1/2}} = \lim_{t\to\infty} t^2(\log t)\left[\Delta v - V(|x|)v\right]\Big|_{|x|=\nu t^{1/2}} \\ &= \lim_{t\to\infty} t^2(\log t)(\Delta v)(x,t)\Big|_{|x|=\nu t^{1/2}} = \lim_{s\to\infty} s(\Delta w)(y,s)\Big|_{|y|=\nu} = 2a_0(\Delta\varphi_0)(y)\Big|_{|y|=\nu} \\ &= -2a_0c_0 + O(\nu^2). \end{aligned}$$

This together with (3.16) and arbitrariness of ν implies that

$$\lim_{t\to\infty} \frac{1}{2}t^2(\log t)^2\zeta'(t) = -2a_0c_0,$$

which gives (3.3) with $l = 1$ for the case $(N, \omega, k) = (2, 0, 0)$. Furthermore, similarly to in the case $(N, \omega, k) \neq (2, 0, 0)$, we have (3.3) with $l = 0$, and the proof of (3.3) for the case $(N, \omega, k) = (2, 0, 0)$ is complete. Therefore, Lemma 3 follows. □

Next we consider the case where $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$. Then, by (2.8) we have

$$P_{k,i}(x) := d_k^{-1}U_k(|x|)Q_{k,i}\left(\frac{x}{|x|}\right) = |x|^k Q_{k,i}\left(\frac{x}{|x|}\right), \qquad |x| < R, \tag{3.17}$$

for $k = 0, 1, 2, \ldots$ and $i \in \{1, \ldots, l_k\}$, and see that $P_{k,i}(x)$ is a homogeneous harmonic polynomial of degree k. Then, applying Lemma 3, we have:

Lemma 4 *Assume the same conditions as in Lemma 3 and let $\zeta = \zeta(t)$ be a function given in Lemma 3. Assume that $V(r) = 0$ in $[0, R]$ for some $R \in (0, \infty)$. Put*

$$\tilde{u}(x,t) = v(x,t)Q_{k,i}\left(\frac{x}{|x|}\right).$$

Then, for any $l \in \{0, 1, 2\}$, there holds

$$\begin{aligned} &\left(\nabla_x^l \tilde{u}\right)(x,t) - d_k\nabla_x^l\left[\zeta(t)P_{k,i}(x) + \zeta'(t)\frac{|x|^2}{2(2k+N)}P_{k,i}(x)\right] \\ &= \begin{cases} O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_k)-2}|x|^{k+4-l}\right) & \text{if } (N,\omega,k) \neq (2,0,0), \\ O\left(t^{-3}[\log t]^{-2}|x|^{4-l}\right) & \text{if } (N,\omega,k) = (2,0,0), \end{cases} \end{aligned} \tag{3.18}$$

for all $x \in B(0, R)$ and sufficiently large t, where d_k is the constant given in (2.8).

Proof By (2.8) and (2.9) we have

$$U_k(r) = d_k r^k, \qquad F_k[U_k](r) = \frac{d_k r^{k+2}}{2(2k+N)}, \tag{3.19}$$

for $r \in [0, R]$. On the other hand, for any $l \in \{0, 1, 2\}$, we have

$$\left|\nabla_x^l Q_{k,i}\left(\frac{x}{|x|}\right)\right| \le C|x|^{-l}, \qquad x \in \mathbf{R}^N,$$

for some constant C. This together with Lemma 3, (3.17), and (3.19) yields (3.18), and the proof of Lemma 4 is complete. □

4 Proof of theorems

Let $\varphi \in L^2(\mathbf{R}^2, \rho\mathrm{d}x)$ and $u \in X$ be a solution of (1.1) under condition (V). By the same argument as in [10], we have radial functions $\{\phi_{k,i}\} \subset L^2(\mathbf{R}^2, \rho\mathrm{d}x)$ satisfying

$$\varphi = \sum_{k=0}^{\infty}\sum_{i=1}^{l_k} \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\mathbf{R}^2, \rho\mathrm{d}x). \tag{4.1}$$

Then, under assumption (1.13), by the orthonormality of $\{Q_{k,i}\}$, (1.7), (1.11), and (4.1) we have

$$M = \int_{\mathbf{R}^N} \varphi(x)\mathcal{U}_{0,1}(x)\mathrm{d}x = q_0 \int_{\mathbf{R}^N} \varphi(x)U_0(|x|)\mathrm{d}x = q_0 \int_{\mathbf{R}^N} \phi_{0,1}(|x|)U_0(|x|)\mathrm{d}x > 0, \tag{4.2}$$

where q_0 is the constant given in (1.9). Furthermore, putting

$$\Phi_{k,i}(x) = \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right),$$

we have

$$\|\varphi\|^2 = \sum_{k=0}^{\infty}\sum_{i=1}^{l_k} \|\Phi_{k,i}\|^2. \tag{4.3}$$

For any $f \in L^2(\mathbf{R}^N, \rho\mathrm{d}x)$, let $S(t)f \in X$ be a solution of (1.1) with the initial function f, and put

$$u_{k,i}(x,t) = (S(t)\Phi_{k,i})(x).$$

Then, we define a function $E_m u$ $(m = 1, 2, \dots)$ by

$$(E_m u)(\cdot, t) := u(\cdot, t) - \sum_{k=0}^{m-1}\sum_{i=1}^{l_k} u_{k,i}(\cdot, t) = S(t)\left[\varphi - \sum_{k=0}^{m-1}\sum_{i=1}^{l_k} \Phi_{k,i}\right]. \tag{4.4}$$

We remark that assertion (i) of Sect. 2.2 implies that

$$u_{k,i}(x,t) = v_{k,i}(x,t)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad \mathbf{R}^N \times (0, \infty), \tag{4.5}$$

where $v_{k,i}(x,t) = [S_k(t)\phi_{k,i}](x)$, and by (1.8) we have

$$\|u_{k,i}(t)\|_2 = q_0\|v_{k,i}(t)\|_2 \tag{4.6}$$

for all $t \geq 0$. Furthermore, by the orthogonality of $\{Q_{k,i}\}$ and the radial symmetry of $v_{k,i}$ we have

$$\int_{\mathbf{R}^N} u_{k,i}(x,t)u_{k',i'}(x,t)\mathrm{d}x = 0 \qquad \text{if} \qquad (k,i) \neq (k',i') \tag{4.7}$$

for all $t > 0$. Then, we have:

Lemma 5 *Assume the same conditions as in Theorem 1. Let* $m = 1, 2, \ldots$. *Then, for any* $L > 0$ *and* $l = 0, 1, 2$, *there exists a positive constant* C *such that*

$$\left|\nabla_x^l (E_m u)(x,t)\right| \leq C t^{-\frac{N}{2}-\alpha(\omega+\omega_m)} \|\varphi\| \tag{4.8}$$

for all $x \in B(0, L)$ *and all sufficiently large* t.

Proof Let $\tilde{m} \in \mathbf{N}$ such that $\tilde{m} \geq m \geq 1$ and $\alpha(\omega + \omega_{\tilde{m}}) \geq 2\alpha(\omega + \omega_m)$. Let $T > 0$. Since it follows from (3.2) that

$$|v_{k,i}(x,t)| = |[S_k(t)\phi_{k,i}](x)| \leq [S_{\tilde{m}}(t)|\phi_{k,i}|](x)$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$, by (4.6) we apply (3.8) to obtain

$$\begin{aligned}\|u_{k,i}(t)\|_2 &= q_0\|v_{k,i}(t)\|_2 \leq C_1 q_0 (1+t)^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_{\tilde{m}})}{2}} \|\phi_{k,i}\| \\ &= C_1(1+t)^{-\frac{N}{4}-\frac{\alpha(\omega+\omega_{\tilde{m}})}{2}} \|\Phi_{k,i}\|\end{aligned}$$

for all $t > 0$, $k \geq \tilde{m}$, and $i = 1, \ldots, l_k$, where C_1 is a constant independent of k and i. This together with (4.1), (4.3), and (4.7) implies that

$$\begin{aligned}\|(E_{\tilde{m}}u)(t)\|_2^2 &= \sum_{k=\tilde{m}}^{\infty}\sum_{i=1}^{l_k} \|u_{k,i}(t)\|_2^2 \\ &\leq C_1 t^{-\frac{N}{2}-\alpha(\omega+\omega_{\tilde{m}})} \sum_{k=\tilde{m}}^{\infty}\sum_{i=1}^{l_k} \|\Phi_{k,i}\|^2 = C_1 t^{-\frac{N}{2}-\alpha(\omega+\omega_{\tilde{m}})} \|\varphi\|^2, \quad t > T.\end{aligned} \tag{4.9}$$

On the other hand, since $(E_{\tilde{m}}u)(t) = S(t/2)[(E_{\tilde{m}}u)(t/2)]$, the comparison principle together with (V) (ii) implies that

$$|(E_{\tilde{m}}u)(x,t)| \leq \left[e^{t\Delta/2}|(E_{\tilde{m}}u)(t/2)|\right](x), \qquad (x,t) \in \mathbf{R}^N \times (0,\infty).$$

This together with the standard L^p–L^q estimate for the heat equation and (4.9) yields

$$\begin{aligned}\|(E_{\tilde{m}}u)(t)\|_\infty &\leq \|e^{t\Delta/2}|(E_{\tilde{m}}u)(t/2)|\|_\infty \\ &\leq C_2 t^{-\frac{N}{4}} \|(E_{\tilde{m}}u)(t/2)\|_2 \leq C_3 t^{-\frac{N}{2}-\frac{\alpha(\omega+\omega_{\tilde{m}})}{2}} \|\varphi\|\end{aligned} \tag{4.10}$$

for all $t > 2T$, where C_2 and C_3 are constants. Therefore, since $\alpha(\omega+\omega_{\tilde m}) \ge 2\alpha(\omega+\omega_m)$, by (3.1), (4.3), (4.4), (4.5), and (4.10), for any sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}|(E_m u)(x,t)| &\le \sum_{k=m}^{\tilde m-1}\sum_{i=1}^{l_k}\left|v_{k,i}(x,t)Q_{k,i}\left(\frac{x}{|x|}\right)\right| + |(E_{\tilde m}u)(x,t)|\\ &\le C_4\sum_{k=m}^{\tilde m-1}\sum_{i=1}^{l_k}\left|v_{k,i}(x)\right| + |(E_{\tilde m}u)(x,t)|\\ &\le C_5 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}U_m(|x|)\sum_{k=m}^{\tilde m-1}\sum_{i=1}^{l_k}\|\phi_{k,i}\| + C_4 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}\|\varphi\|\\ &\le C_6 t^{-\frac{N}{2}-\alpha(\omega+\omega_m)}(1+U_m(|x|))\|\varphi\|\end{aligned}$$

for all $(x,t) \in D_\epsilon(2T)$, where C_4, C_5, and C_6 are constants. This implies inequality (4.8) with $l = 0$. Furthermore, by (4.8) with $l = 0$ we apply the parabolic regularity theorems to obtain inequality (4.8) with $l = 1, 2$. Thus, Lemma 5 follows. □

Now we are ready to prove Theorems 1 and 2.

Proof of Theorem 1 We first prove assertion (a). Let $R > 0$. By Lemma 3 with $k = 0$ and (4.2), we see that there exists a function $\zeta_0 = \zeta_0(t)$ defined in $(0,\infty)$ such that

$$\frac{d^n}{dt^n}\zeta_0(t) = \begin{cases}(cM+o(1))\dfrac{d^n}{dt^n}t^{-\frac{N}{2}-\alpha(\omega)} & \text{if } (N,\omega)\ne(2,0),\\ (4cM+o(1))\dfrac{d^n}{dt^n}[t^{-1}(\log t)^{-2}] & \text{if } (N,\omega)=(2,0),\end{cases} \tag{4.11}$$

and

$$v_{0,1}(x,t) = \zeta_0(t)U_0(r) + \zeta_0'(t)F_0[U_0](r) + F_0\left[F_0[(\partial_t^2 v_{0,1})(\cdot,t)]\right](|x|) \tag{4.12}$$

for all $x \in B(0,R)$ and all sufficiently large t, where $n = 0, 1$ and

$$c = \frac{c_0^2}{M}\int_{\mathbf{R}^N}\phi_{0,1}(x)U_0(|x|)dx = \frac{c_0^2}{q_0} > 0.$$

Then, since $u_{0,1}(x,t) = q_0 v_{0,1}(x,t)$, by (3.5), (4.11), and (4.12) we have

$$\begin{aligned}&(\nabla_x^l u_{0,1})(x,t) - q_0\nabla_x^l\left[\zeta_0(t)U_0(r)+\zeta_0'(t)F_0[U_0](r)\right]\Big|_{r=|x|}\\ &= \begin{cases}O(t^{-\frac{N}{2}-\alpha(\omega)-2}|x|^{4-l}) & \text{if } (N,\omega)\ne(2,0),\\ O(t^{-3}(\log t)^{-2}|x|^{4-l}) & \text{if } (N,\omega)=(2,0),\end{cases}\end{aligned} \tag{4.13}$$

for all $x \in B(0,R)$ and all sufficiently large t, where $l = 0, 1, 2$, Then, since $cM > 0$, (4.11) and (4.13) imply

$$\zeta_0(t) > 0, \qquad \zeta_0'(t) < 0, \qquad \zeta_0'(t) = O(t^{-1}\zeta_0(t)), \tag{4.14}$$

$$\left(\nabla_x^l u_{0,1}\right)(x,t) - q_0\nabla_x^l\left[\zeta_0(t)U_0(r)+\zeta_0'(t)F_0[U_0](r)\right]\Big|_{r=|x|} = O\left(t^{-1}|\zeta_0'(t)||x|^{4-l}\right) \tag{4.15}$$

for all $x \in B(0, R)$ and all sufficiently large t, where $l = 0, 1, 2$. Therefore, by (4.14), (4.15), and Lemma 5, we have

$$\left(\nabla_x^l u\right)(x, t) - q_0 \nabla_x^l \left[\zeta_0(t) U_0(|x|) + \zeta_0'(t) F_0[U_0](|x|)\right]$$
$$= O\left(t^{-1}|\zeta_0'(t)|\right) + \left(\nabla_x^l E_1 u\right)(x, t) = O\left(t^{-1}|\zeta_0'(t)|\right) + O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}\right) \tag{4.16}$$

for all $x \in B(0, R)$ and all sufficiently large t. In particular, by (2.6) and (4.14) we have

$$u(x, t) = q_0 \zeta_0(t) U_0(r) + O\left(t^{-1}\zeta_0(t)\right) + O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}\right) \geq \frac{q_0 d_0}{2} \zeta_0(t) > 0$$

for all $x \in B(0, R)$ and all sufficiently large t. This gives assertion (a).

Next we prove assertion (b) of Theorem 1. Assume that $V(r) \not\equiv 0$ on $[0, R]$ for some $R > 0$, and let $r_0 \in (0, R)$ such that $V(r_0) > 0$. Then, by (2.4) and (2.6), we see that there exist positive constants δ and η such that

$$U_0'(r) = r^{1-N} \int_0^r \tau^{N-1} V(\tau) U_0(\tau) \mathrm{d}\tau \geq \eta > 0$$

for all $r \in [r_0 - \delta, r_0 + \delta] \subset (0, R)$. This together with (4.11), (4.14), and (4.16) implies that

$$(\partial_r u)(x, t) = q_0 \zeta_0(t) U_0'(|x|) + O(t^{-1}\zeta_0(t)) + O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_1)}\right) \geq \frac{q_0 \eta}{2} \zeta_0(t) > 0 \tag{4.17}$$

for all $x \in \mathbf{R}^N$ with $r_0 - \delta \leq |x| \leq r_0 + \delta < R$ and all sufficiently large t. Put

$$\lambda(t) := u_{0,1}(x, t)\Big|_{|x|=r_0} = q_0 v_{0,1}(x, t)\Big|_{|x|=r_0}.$$

By Lemma 5 and (4.11), we have

$$u(x, t)\Big|_{|x|=r_0} - \lambda(t) = (E_1 u)(x, t)\Big|_{|x|=r_0} = o(\zeta_0(t)).$$

This together with (4.17) implies that, for any $\omega \in \mathbf{S}^{N-1}$ and any sufficiently large t, there exists a constant $r(\omega, t) \in (r_0 - \delta, r_0 + \delta)$ such that

$$u(r(\omega, t)\omega, t) = \lambda(t).$$

Then, by (4.17) we have

$$\left\{x \in \mathbf{R}^N : r\left(\frac{x}{|x|}, t\right) < |x| < r_0 + \delta\right\} \subset \{x \in B(0, R) : u(x, t) > \lambda(t)\}$$

and

$$\left\{x \in \mathbf{R}^N : r_0 - \delta < |x| < r\left(\frac{x}{|x|}, t\right)\right\} \cap \{x \in B(0, R) : u(x, t) > \lambda(t)\} = \emptyset.$$

These imply that the function $u(\cdot, t)$ is not quasi-concave in $B(0, R)$ for any sufficiently large t. Thus, we obtain assertion (b).

It remains to prove assertion (c). Let $V(r) \equiv 0$ on $[0, R]$ for some $R > 0$, and assume $\omega < \omega_*$. Let $\alpha, \beta \in \{1, \dots, N\}$, and put $\partial_\alpha := \partial/\partial x_\alpha$. Then, by (1.7) we have

$$P_{0,1}(x) = q_0, \qquad \partial_\alpha \partial_\beta P_{1,i}(x) = 0 \quad (i = 1, \dots, N),$$

and by Lemma 4 and (4.11) we obtain

$$(\partial_\alpha \partial_\beta u_{0,1})(x,t) = \frac{d_0 q_0}{N} \zeta_0'(t) \delta_{\alpha\beta} + O\left(t^{-1} |\zeta_0'(t)|\right), \tag{4.18}$$

$$(\partial_\alpha \partial_\beta u_{1,i})(x,t) = O\left(t^{-\frac{N}{2} - \alpha(\omega+\omega_1) - 1}\right), \qquad i = 1, \dots N, \tag{4.19}$$

for all $x \in B(0, R)$ and all sufficiently large t, where $i \in \{1, \dots, N\}$. On the other hand, by Lemma 1 we have

$$\alpha(\omega + \omega_2) > \alpha(\omega) + 1. \tag{4.20}$$

Then, by Lemma 5, (4.11), and (4.18)–(4.20) we have

$$\begin{aligned}
\partial_\alpha \partial_\beta u(x,t) &= (\partial_\alpha \partial_\beta u_{0,1})(x,t) + \sum_{i=1}^N (\partial_\alpha \partial_\beta u_{1,i})(x,t) + (\partial_\alpha \partial_\beta E_2 u)(x,t) \\
&= \frac{d_0 q_0}{N} \zeta_0'(t) \delta_{\alpha\beta} + O\left(t^{-1} |\zeta_0'(t)|\right) + O\left(t^{-\frac{N}{2} - \alpha(\omega+\omega_1) - 1}\right) + O\left(t^{-\frac{N}{2} - \alpha(\omega+\omega_2)}\right) \\
&= \frac{d_0 q_0}{N} \zeta_0'(t) \delta_{\alpha\beta} + o(|\zeta_0'(t)|)
\end{aligned} \tag{4.21}$$

for all $x \in B(0, R)$ and all sufficiently large t. Since $\zeta_0'(t) < 0$ by (4.14), (4.21) implies that $u(\cdot, t)$ is concave in $B(0, R)$ for all sufficiently large t. Therefore, we have assertion (c) of Theorem 1, and the proof of Theorem 1 is complete. □

Proof of Theorem 2 Let ϕ be a smooth function in $[0, \infty)$ such that $\operatorname{supp}\phi \subset (1/2, 1)$ and $\phi \geq (\not\equiv)\, 0$ in $[0, \infty)$. Without loss of generality, we can assume that

$$Q_{2,1}\left(\frac{x}{|x|}\right) = q_2 \frac{x_1^2 - x_2^2}{|x|^2} \tag{4.22}$$

for some positive constant q_2, and by (3.17) we have

$$P_{2,1}(x) := |x|^2 Q_{2,1}\left(\frac{x}{|x|}\right) = q_2 \left(x_1^2 - x_2^2\right).$$

For any $h > 0$, put

$$\varphi(x) = \phi(|x|) + q_2^{-1} h \phi(|x|) \left(x_1^2 - x_2^2\right), \qquad x \in \mathbf{R}^N.$$

Then, defining $\phi_{k,i}$ as in (4.1), we see that

$$\phi_{k,i}(|x|) = \begin{cases} \phi(|x|) & \text{for} \quad (k,i) = (0,1), \\ h\phi(|x|)|x|^2 & \text{for} \quad (k,i) = (2,1), \\ 0 & \text{otherwise}, \end{cases} \tag{4.23}$$

and have

$$M \equiv q_0 \int_{\mathbf{R}^N} \phi_{0,1}(|x|) U_0(|x|) \mathrm{d}x > 0, \qquad M' := \int_{\mathbf{R}^N} \phi_{2,1}(|x|) U_2(|x|) \mathrm{d}x > 0. \tag{4.24}$$

Furthermore, by (4.23) we have

$$u_{1,i}(x,t) = 0 \quad (i = 1, \ldots, N) \quad \text{and} \quad (E_2 u)(x,t) = u_{2,1}(x,t). \tag{4.25}$$

Then, by (2.8) and (4.24) we apply Lemma 4 to $u_{2,1}$ to obtain

$$\left(\partial_1^2 E_2 u\right)(x,t) = \left(\partial_1^2 u_{2,1}\right)(x,t) = C_1 M' t^{-\frac{N}{2}-\alpha(\omega+\omega_2)} + O\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_2)-1}\right) \tag{4.26}$$

for all $x \in B(0,R)$ and all sufficiently large t, where C_1 is a positive constant.

We first prove assertion (a). Since $\omega > \omega_*$ or $(N,\omega) = (2,\omega_*)$, by (4.11) we have

$$\zeta_0'(t) = o\left(t^{-\frac{N}{2}-\alpha(\omega+\omega_2)}\right)$$

as $t \to \infty$. Then, by (4.21), (4.25), and (4.26) we have

$$(\partial_1 \partial_1 u)(x,t) \geq \frac{C_1 M'}{2} t^{-\frac{N}{2}-\alpha(\omega+\omega_2)} > 0 \tag{4.27}$$

for all $x \in B(0,R)$ and all sufficiently large t. This implies that $u(\cdot,t)$ is not quasi-concave in $B(0,R)$ for all sufficiently large t. Furthermore, taking a sufficiently small $h > 0$ so that $\varphi \geq 0$ in $\mathbf{R}^N$ if necessary, we see that $u(x,t) > 0$ in $\mathbf{R}^N \times (0,\infty)$, and obtain assertion (a).

Next we prove assertion (b). Since $\alpha(\omega) + 1 = \alpha(\omega + \omega_2)$, by (4.11) and (4.24) we see that there exists a positive constant C_2 such that

$$\zeta_0'(t) = -(C_2 M + o(1)) t^{-\frac{N}{2}-\alpha(\omega)-1} = -(C_2 M + o(1)) t^{-\frac{N}{2}-\alpha(\omega+\omega_2)}$$

as $t \to \infty$. Then, by (4.18), (4.21), (4.25), and (4.26) we have

$$(\partial_1^2 u)(x,t) = \left(-\frac{d_0 q_0 C_2 M}{N} + C_1 M' + o(1)\right) t^{-\frac{N}{2}-\alpha(\omega+\omega_2)} \tag{4.28}$$

for all $x \in B(0,R)$ and all sufficiently large t. On the other hand, by (4.24) we can take a sufficiently large h so that

$$-\frac{d_0 q_0 C_2 M}{N} + C_1 M' > 0.$$

Then, by (4.28) we have

$$(\partial_1^2 u)(x,t) > 0$$

for all $x \in B(0,R)$ and all sufficiently large t. Therefore, $u(\cdot,t)$ is not quasi-concave in $B(0,R)$ for all sufficiently large t, and by (4.24) we have assertion (b). Thus, Theorem 2 follows. □

References

1. Borell, C.: Brownian motion in a convex ring and quasiconcavity. Commun. Math. Phys. **86**, 143–147 (1982)
2. Borell, C.: Hitting probabilities of killed Brownian motion: a study on geometric regularity. Ann. Sci. École Norm. Sup. **17**, 51–467 (1984)
3. Borell, C.: Greenian potentials and concavity. Math. Ann. **272**, 155–160 (1985)
4. Borell, C.: Geometric properties of some familiar diffusions in $\mathbf{R}^n$. Ann. Probab. **21**, 482–489 (1993)
5. Borell, C.: A note on parabolic convexity and heat conduction. Ann. Inst. H. Poincaré Probab. Statist **32**, 387–393 (1996)

6. Brascamp, H.J., Lieb, E.H.: On the extensions of the Brunn–Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Func. Anal. **22**, 366–389 (1976)
7. Diaz, J.I., Kawohl, B.: On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings. J. Math. Anal. Appl. **177**, 263–286 (1993)
8. Giga, Y., Goto, S., Ishi, H., Sato, M.-H.: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J. **40**, 443–470 (1991)
9. Greco, A., Kawohl, B.: Log-concavity in some parabolic problems. Electron. J. Differ. Equ. **1999**, 1–12 (1999)
10. Ishige, K., Kabeya, Y.: Large time behaviors of hot spots for the heat equation with a potential. J. Differ. Equ. **244** 2934–2962 (2008); Corrigendum in J. Differ. Equ. **245**, 2352–2354 (2008)
11. Ishige, K., Kabeya, Y.: Hot spots for the heat equation with a rapidly decaying negative potential. Adv. Differ. Equ. **14**, 643–662 (2009)
12. Ishige, K., Kabeya, Y.: Hot spots for the two dimensional heat equation with a rapidly decaying negative potential. Discrete Contin. Dyn. Syst. Ser. S **4**, 833–849 (2011)
13. Ishige, K., Kabeya, Y.: L^p norms of nonnegative Schrödinger heat semigroup and the large time behavior of hot spots, preprint
14. Ishige, K., Murata, M.: An intrinsic metric approach to uniqueness of the positive Cauchy problem for parabolic equations. Math. Z. **227**, 313–335 (1998)
15. Ishige, K., Murata, M.: Uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on manifolds or domains. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **30**, 171–223 (2001)
16. Ishige, K., Salani, P.: Is quasi-concavity preserved by heat flow? Arch. Math. **90**, 450–460 (2008)
17. Ishige, K., Salani, P.: Convexity breaking of the free boundary for porous medium equations. Interfaces Free Bound **12**, 75–84 (2010)
18. Ishige, K., Salani, P.: Parabolic quasi-concavity for solutions to parabolic problems in convex rings. Math. Nachr. **283**, 1526–1548 (2010)
19. Ishige, K., Salani, P.: On a new kind of convexity for solutions of parabolic problems. Discrete Contin. Dyn. Syst. Ser. S **4**, 851–864 (2011)
20. Janson, S., Tysk, J.: Preservation of convexity of solutions to parabolic equations. J. Differ. Equ. **206**, 182–226 (2004)
21. Kennington, A.U.: Power concavity and boundary value problems. Indiana Univ. Math. J. **34**, 687–704 (1985)
22. Korevaar, N.J.: Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. **32**, 603–614 (1983)
23. Lee, K.-A.: Power concavity on nonlinear parabolic flows. Commun. Pure Appl. Math. **58**, 1529–1543 (2005)
24. Lee, K.-A., Vázquez, J.L.: Geometrical properties of solutions of the porous medium equation for large times. Indiana Univ. Math. J. **52**, 991–1016 (2003)
25. Lee, K.-A., Petrosyan, A., Vázquez, J.L.: Large-time geometric properties of solutions of the evolution p-Laplacian equation. J. Differ. Equ. **229**, 389–411 (2006)
26. Lions, P.-L., Musiela, M.: Convexity of solutions of parabolic equations. C. R. Math. Acad. Sci. Paris **342**, 915–921 (2006)