Green's Function for the 1D Poisson Equation

John McCuan

April 7, 2020

1 The Classical Problem and Green's Function

Here we consider the ODE

$$-u'' = f \qquad \text{(1D Poisson Equation)}$$

and the boundary value problem

$$\begin{cases} -u'' = f & \text{on } (0, L) \\ u(0) = u(L) = 0 \end{cases} \tag{1}$$

where f is classically a continuous function on $[a, b]$ but may have considerably relaxed regularity in the discussion below and $L > 0$. We have observed that

$$G(x, \xi) = \begin{cases} (L - \xi)x/L, & 0 \le x \le \xi \\ -\xi(x - L)/L, & \xi \le x \le L \end{cases} \tag{2}$$

is, for each fixed $\xi \in (0, L)$, a Lipschitz function with

$$u(x) = \int_0^L G(x, \xi) f(\xi)\, d\xi$$

a classical solution of (1) when $f \in C^0[0, L]$.

2 The Definition of a Distributional Solution

We wish to examine the sense in which $u(x) = G(x, \xi)$ is a solution of

$$\begin{cases} \text{“}-u'' = \delta_\xi\text{”} & \text{on } (0, L) \\ u(0) = u(L) = 0 \end{cases} \tag{3}$$

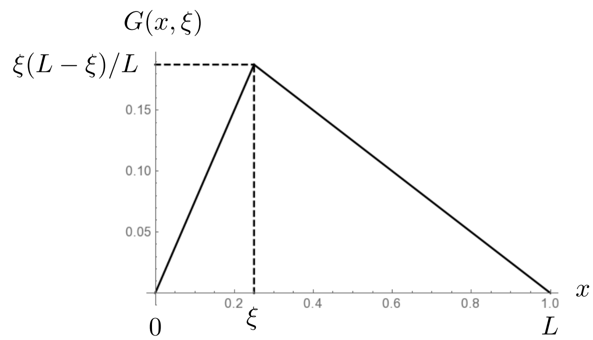

Figure 1: The 1D Green's function for the Laplace operator on the interval $[0, L]$.

where δ_ξ is the Dirac delta distribution at ξ, that is $\delta_\xi : C_c^\infty(0, L) \to \mathbb{R}$ by

$$\delta_\xi[\phi] = \phi(\xi).$$

We interpret the PDE in (3) in two ways applicable to $u(x) = G(x, \xi)$. The first way is

$$-\int_0^L u(x)\phi''(x)\,dx = \delta_\xi[\phi] = \phi(\xi) \qquad \text{for every } \phi \in C_c^\infty(0, L). \tag{4}$$

The second equivalent way is

$$\int_0^L u'(x)\phi'(x)\,dx = \delta_\xi[\phi] = \phi(\xi) \qquad \text{for every } \phi \in C_c^\infty(0, L) \tag{5}$$

where $u' = G_x$ is a weak derivative of u. We have verified that $G = G(x, \xi)$ has a weak derivative given by

$$G_x(x,\xi) = \begin{cases} (L-\xi)/L, & 0 \le x < \xi \\ -\xi/L, & \xi < x \le L. \end{cases} \tag{6}$$

The value of $G_x(x, \xi)$ is not specified at $x = \xi$, but G_x need only be defined as a function in L^1_{loc}, so this is no problem. Moreover, we have verified that $u(x) = G(x, \xi)$ satisfies both "distributional" formulations (4) and (5).

Exercise 1 *Verify that G_x given by (6) is a weak derivative of G and that both (4) and (5) are satisfied by $u(x) = G(x, \xi)$.*

3 The Meaning of the Solution

We now seek to understand the meaning of these formulations in terms of approximation by (non-distributional) functions and weak solutions in particular. We observe that

$$\mu = \mu_\delta(x) = \frac{1}{2\delta}\chi_{[\xi-\delta,\xi+\delta]}(x) = \begin{cases} 1/(2\delta), & |x-\xi| \le \delta \\ 0, & |x-\xi| > \delta \end{cases} \tag{7}$$

considered as a distribution $M : C_c^\infty(0,L) \to \mathbb{R}$ by

$$M[\phi] = \int \mu\phi$$

satisfies

$$\lim_{\delta \searrow 0} M[\phi] = \delta_\xi[\phi] = \phi(\xi) \qquad \text{for all } \phi \in C_c^\infty(0,L).$$

Thus, $M = M_\delta$ approaches the Dirac delta distribution as the positive parameter δ tends to 0. The graph of the function μ is illustrated in Figure 2.

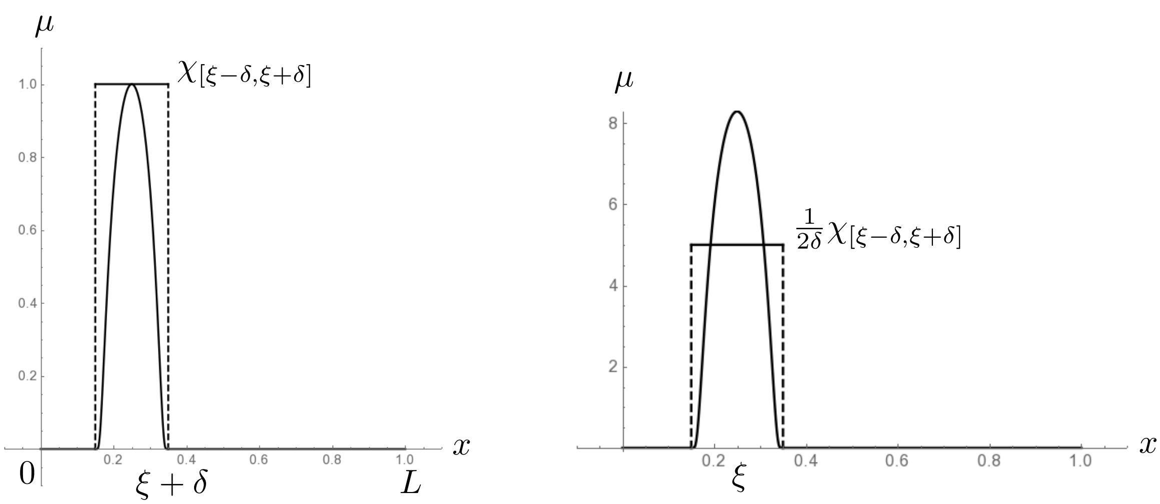

Figure 2: Test functions with support in the interval $[\xi - \delta, \xi + \delta] \subset [0, L]$: The characteristic function of height 1 and a smooth test function of max height 1 are illustrated on the left. On the right these functions are normalized (i.e., scaled) to have $\int \mu = 1$. Note the figure on the right uses different scales for the axes; the height of the smooth μ is about 8.286 with $\delta = 0.1$.

We propose to consider the approximating problem

$$\begin{cases} -u'' = \mu & \text{on } (0, L) \\ u(0) = u(L) = 0 \end{cases} \tag{8}$$

with the expectation that the solution $u = u_\delta$ approaches the Green's function as $\delta \searrow 0$. This will illustrate the meaning of having a **unit point source** at ξ. Each of the forcing functions $\mu = \mu_\delta$ is of unit magnitude in $L^1(0, L)$:

$$\|\mu\|_{L^1} = \int |\mu| = 1.$$

This problem does not admit a classical solution, though the regularity does allow a weak formulation of the problem (rather than a strictly distributional one). The added regularity will be manifest in that the solution we find will be $C^1[0, L]$ in contrast to the Green's function which is only Lipschitz. Nevertheless, the classical second derivative of $u = u_\delta$ will not be defined at the two points $x = \xi \pm \delta$, so we must turn to a weak formulation of (8). As in the case of the distributional problem for the Green's function, we can write down two equivalent (weak) formulations:

$$-\int_0^L u(x)\phi''(x)\,dx = \int_{(0,L)} \mu\phi \qquad \text{for every } \phi \in C_c^\infty(0, L). \tag{9}$$

The second equivalent way is

$$\int_0^L u'(x)\phi'(x)\,dx = \int_{(0,L)} \mu\phi. \qquad \text{for every } \phi \in C_c^\infty(0, L) \tag{10}$$

The second formulation assumes u has a weak derivative $u' \in L^1_{loc}(0, L)$. Let us postulate a piecewise classical solution of the form

$$u(x) = \begin{cases} c_1 x, & 0 \le x \le \xi - \delta \\ -(x - \xi)^2/(4\delta) + a(x - \xi) + b, & \xi - \delta \le x \le \xi + \delta \\ c_2(x - L), & \xi + \delta \le x \le L. \end{cases} \tag{11}$$

This function u will be continuous if $u((\xi - \delta)^-) = c_1(\xi - \delta) = u((\xi - \delta)^+)$ and $u((\xi + \delta)^-) = u((\xi + \delta)^+)$. That is,

$$c_1(\xi - \delta) = -(-\delta)^2/(4\delta) + a(-\delta) + b, \text{ and}$$
$$c_2(\xi + \delta - L) = -(\delta)^2/(4\delta) + a\delta + b,$$

or

$$\begin{cases} 4c_1(\xi - \delta) = 4b - 4\delta a - \delta \text{ and} \\ 4c_2(\xi + \delta - L) = 4b + 4\delta a - \delta. \end{cases} \tag{12}$$

Exercise 2 *Show that if (12) holds, then*

$$u'(x) = \begin{cases} c_1, & 0 \le x < \xi - \delta \\ -(x-\xi)/(2\delta) + a, & \xi - \delta \le x \le \xi + \delta \\ c_2, & \xi + \delta \le x \le L \end{cases} \tag{13}$$

is a weak derivative of the function $u = u_\delta$ given by (11).

In view of this result, we turn to the weak formulation of (10) according to which we need

$$c_1 \int_0^{\xi-\delta} \phi'(x)\, dx + \int_{\xi-\delta}^{\xi+\delta} \left[-\frac{1}{2\delta}(x-\xi) + a \right] \phi'(x)\, dx + c_2 \int_{\xi+\delta}^{L} \phi'(x)\, dx$$
$$= \int_{\xi-\delta}^{\xi+\delta} \mu(x)\phi(x)\, dx.$$

This simplifies to

$$c_1\phi(\xi-\delta) - \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} (x-\xi)\phi'(x)\, dx + a\left[\phi(\xi+\delta) - \phi(\xi-\delta)\right] - c_2\phi(\xi+\delta)$$
$$= \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} \phi(x)\, dx.$$

Notice that the right side is the average value of ϕ over the interval $(\xi-\delta, \xi+\delta)$ which, for a smooth function ϕ will tend to $\delta_\xi[\phi] = \phi(\xi)$ as $\delta \searrow 0$. This is an interesting observation we will not need to use here, but it is worth being able to prove.

Exercise 3 *Show that for any smooth function $\phi \in C_c^\infty(\mathbb{R})$ one has*

$$\lim_{\delta \searrow 0} \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} \phi(x)\, dx = \phi(\xi).$$

Turning to the remaining integral on the left, we integrate by parts to find

$$\int_{\xi-\delta}^{\xi+\delta} (x-\xi)\phi'(x)\, dx = (x-\xi)\phi(x)\Big|_{\xi-\delta}^{\xi+\delta} - \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, dx$$
$$= \delta\phi(\xi+\delta) + \delta\phi(\xi-\delta) - \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, dx.$$

Substituting this calculation into the weak formulation above and simplifying, we find the average values of ϕ cancel and the condition becomes

$$\left(c_1 - a - \frac{1}{2}\right)\phi(\xi - \delta) = \left(c_2 - a + \frac{1}{2}\right)\phi(\xi + \delta).$$

Now we choose some specific test functions ϕ to obtain information from this condition. First we take $\phi \in C_c^\infty(\xi - 2\delta, \xi) \subset C_c^\infty(0, L)$ with $\phi(\xi - \delta) \neq 0$. See Figure 3. It

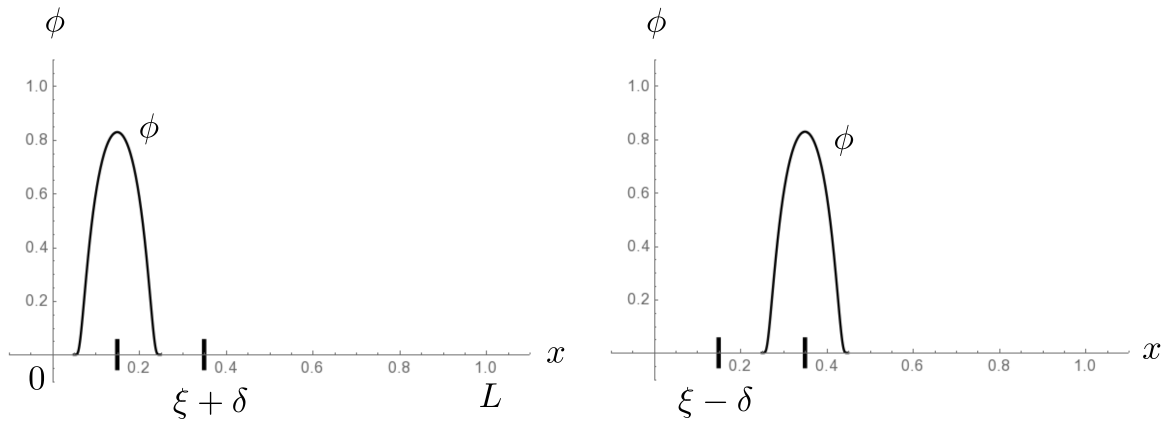

Figure 3: Test functions with support centered at $\xi - \delta$ (left) and support centered at $\xi + \delta$ (right).

follows from this choice that for u to be a weak solution of the ODE, we must have

$$c_1 = a + \frac{1}{2}.$$

Taking a test function ϕ with support shifted right so that $\phi(\xi+\delta) \neq 0$ but $\phi(\xi-\delta) = 0$, we have also

$$c_2 = a - \frac{1}{2}.$$

We may now substitute these values for c_1 and c_2 into the system (12) obtained from the requirement that $u \in C^0[0, L]$. This gives

$$\begin{cases} 4\,(a + 1/2)\,(\xi - \delta) = 4b - 4\delta a - \delta \\ 4\,(a - 1/2)\,(\xi + \delta - L) = 4b + 4\delta a - \delta \end{cases}$$

or

$$\begin{cases} 4\xi a - 4b &= \delta - 2\xi \\ 4(\xi - L)a - 4b &= \delta + 2\xi - 2L. \end{cases} \tag{14}$$

Thus, we see the only possible continuous solution of the form (11) has

$$\begin{aligned} a &= \frac{1}{4L}(2L - 4\xi) = \frac{L - 2\xi}{2L} = \frac{1}{2} - \frac{\xi}{L} \\ b &= \frac{1}{4}\left(4\xi - \frac{4}{L}\xi^2 - \delta\right) = \xi - \frac{\xi^2}{L} - \frac{\delta}{4} \\ c_1 &= 1 - \frac{\xi}{L} = \frac{L - \xi}{L} \\ c_2 &= -\frac{\xi}{L}. \end{aligned}$$

Thus, we have obtained a weak solution of the approximating boundary value problem (8) given by

$$u(x) = \begin{cases} (L - \xi)x/L, & 0 \le x \le \xi - \delta \\ -(x - \xi)^2/(4\delta) + \frac{L-2\xi}{2L}(x - \xi) - \frac{4\xi^2 - 4L\xi + \delta L}{4L}, & \xi - \delta \le x \le \xi + \delta \\ -\xi(x - L)/L, & \xi + \delta \le x \le L. \end{cases}$$

One notes at this point the striking comparison with the Green's function (2); our solution matches the Green's function identically on the intervals $[0, \xi - \delta]$ and $[\xi + \delta, L]$. The quadratic value of $u = u_\delta$ on $[\xi - \delta, \xi + \delta]$ may also be written as

$$-\frac{1}{4\delta}x^2 + \left(\frac{\xi}{2\delta} + \frac{1}{2} + \frac{\xi}{L}\right)x - \frac{\xi^2}{4\delta} - \frac{1}{2\xi} + \xi - \frac{\delta}{4}$$

though no special insight on this quadratic part seems to be gained from doing so. From either expression it may be checked that $u'((\xi - \delta)^+) = (L - \xi)/L$ and $u'((\xi + \delta)^-) = -\xi/L$, so that $u \in C^1[0, L]$. More generally,

$$u'(x) = \begin{cases} (L - \xi)/L, & 0 \le x \le \xi - \delta \\ -(x - \xi)/(2\delta) + (L - 2\xi)/(2L), & \xi - \delta \le x \le \xi + \delta \\ -\xi/L, & \xi + \delta \le x \le L. \end{cases}$$

Since a C^1 function is clearly an $H^1 = W^{1,2}$ function, we have found a weak solution in $H^1(0, L)$. In fact, this solution satisfies the boundary conditions classically and it can be fairly easily seen[1] that $u \in H^1_0(0, L)$.

[1] In fact, we will attempt to demonstrate this explicitly in connection with some other considerations below.

Exercise 4 *Show that* $u = u_\delta \notin W^2(0,L)$, *i.e.,* $u' \notin W^1_{loc}(0,L)$. *Hint:* u' *has a classical derivative on* $[0,L]\backslash\{\xi \pm \delta\}$.

We know (or will soon prove) that such a solution is known to exist and, more importantly for the present considerations, is **unique**. Thus, we have found the unique weak solution in $H^1_0(0,L)$ for this problem. In regard to regularity, the solution we have found satisfies, in fact,

$$u = u_\delta \in \sqsubset^2[0,L]$$

where $\sqsubset^2[0,L]$ denotes the collection of piecewise C^2 functions on $[0,L]$.

Finally, it is abundantly clear, as suspected, that $u = u_\delta$ tends to $G(x,\xi)$ as δ tends to zero. We have plotted u for some specific choices of the constants in Figure 4. The convergence, for example, is clearly uniform on $[0.L]$, from which it follows that

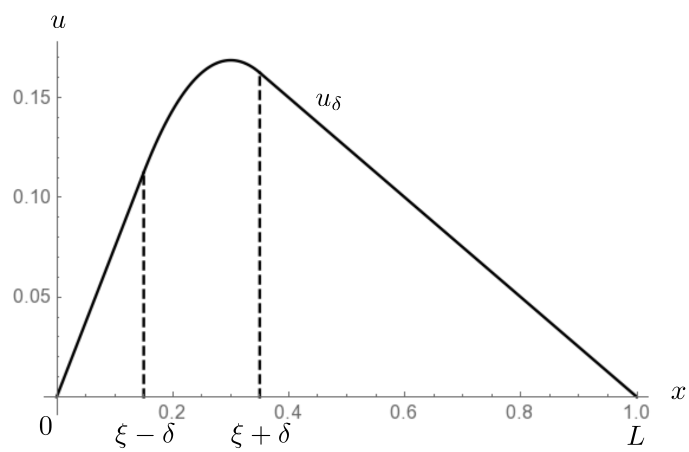

Figure 4: The weak C^1 solution of (8).

$u_\delta \to G$ in L^p for every p. We cannot say $u_\delta \to G$ in H^1 because $G \notin H^1(0,L)$.

4 Comment on the Uniqueness

We have mentioned that the problem (8) has a unique weak solution in $H^1_0(0,L)$. Aside from the minor detail of showing $u \in H^1_0(0,L)$, which we will address below, if one looks over the proof of existence and uniqueness for this problem using the Riesz representation theorem, one notes that everything hinges (in some sense) on verification of a Poincaré inequality appropriate to the dimension under consideration. I have given an argument for the case $n = 2$ of $\mathbb{R}^2$ (the plane) elsewhere, and for general dimensions $n > 2$, one has recourse to the Gagliardo-Nirenberg-Sobolev inequality. I'm going to give quickly a proof of the Poincaré inequality for $C_c^\infty(\mathbb{R})$ specifically required here in regard to $H_0^1(0, L)$.

Theorem 1 *(Poincaré inequality for $\mathbb{R}^1$) Given any open bounded set $U \subset \mathbb{R}$ and any $\phi \in C_c^\infty(U)$, one has*

$$\|\phi\|_{L^2} \leq \mu(U)\|\phi'\|_{L^2}$$

where $\mu(U)$ is the Lebesgue measure U.

Proof: We may assume as usual that ϕ is defined on all of $\mathbb{R}$ by setting $\phi \equiv 0$ for $x \in \mathbb{R}\backslash U$, though we will need to keep in mind the location of the support of ϕ:

$$\operatorname{supp}(\phi) \subset\subset U.$$

By the fundamental theorem of calculus

$$\phi(x) = \int_{-\infty}^{x} \phi'(t)\, dt.$$

This implies

$$|\phi| \leq \int |\phi'|. \tag{15}$$

Squaring and integrating on U we get

$$\int_U |\phi|^2 \leq \int_U \left(\int |\phi'|\right)^2 = \left(\int |\phi'|\right)^2 \int_U 1 = \mu(U)\left(\int |\phi'|\right)^2.$$

Thus, taking the square root

$$\|\phi\|_{L^2} \leq \sqrt{\mu(U)} \int |\phi'| = \sqrt{\mu(U)}\, \|\phi'\|_{L^1}.$$

Continuing from here with the holder inequality since the constant function $\chi_U \equiv 1 \in L^2(U)$, we get

$$\begin{aligned} \|\phi\|_{L^2} &\leq \sqrt{\mu(U)}\langle|\phi'|, \chi_U\rangle_{L^2} \\ &\leq \sqrt{\mu(U)}\|\phi'\|_{L^2}\|\chi_U\|_{L^2} \\ &= \mu(U)\|\phi'\|_{L^2}. \qquad \square \end{aligned}$$

Remark: We recall that the Sobolev conjugate exponents p and p^* are defined by $1/p - 1/p^* = 1/n$. These are usually only used when $n \geq 3$ and the Gagliardo-Nirenberg-Sobolev inequality asserts that

$$\|\phi\|_{L^{p^*}} \leq C\|D\phi\|_{L^p} \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n)$$

with the constant independent of the support of ϕ. In the case $n = 1$ and $p = 1$, note that p^* is formally determined to be $p^* = \infty$. In fact (15) implies

$$\|\phi\|_{L^\infty} \leq \|\phi'\|_{L^1}$$

with the universal constant $C = 1$.

5 Physical Interpretation/Modeling

The function μ from (7) may be interpreted as an external generation or introduction of heat energy at a constant unit rate uniformly along the portion $[\xi - \delta, \xi + \delta]$ of the interval of definition $[0, L]$ which models a "thin heat conducting rod." To further understand the meaning of the forcing term μ and the equation $-u'' = \mu$ in this connection, we briefly discuss the relevant physical quantities involved and the usual derivation of the **1D heat equation** as a model for heat energy diffusion in a rod.

Initial Quantities and Units

We begin with a **heat** (or thermal) **energy density** $\theta_1 = \theta_1(x, t)$ with units

$$[\theta_1] = \frac{[energy]}{L}.$$

The general consideration of units in this connection as well as a further discussion of densities in relation to the modeling of mass and other quantities in various dimensions may be found in the notes on integration. This energy density gives **the spatial density of energy at a point x at time t during the evolution**, so that at any time t, the total energy associated with a subset of the model continuum $[0, L]$ may be calculated from an integral.

Let us also slightly generalize our model by assuming coordinates in the rod on an arbitrary interval $I = [a, b] \subset \mathbb{R}$, so that at any given time t, the total energy within the rod is assumed to be given by the integral

$$\int_{(a,b)} \theta_1 = \int_a^b \theta_1(x, t)\, dx,$$

and the energy between any two points x_1 and x_2 is modeled by

$$\int_{x\in(x_1,x_2)} \theta_1.$$

Note that one may assume θ_1 has minimal regularity, say $L^1(a,b)$, if the integration on the left (suggesting, for example, Lebesgue integration) is used. The expression on the right, more or less, suggests the requirement that θ_1 is continuous. Generally, one considers the distribution of heat energy described by θ_1 to be evolving and unknown.

The next quantity of interest is the **thermal energy flux field** which we may take as a scalar function $\phi_1 = \phi_1(x,t)$ in this case with units

$$[\phi_1] = \frac{[energy]}{T}.$$

The field ϕ_1 gives **the rate at which energy flows across the point** x **at time** t. Thus, the rate at which energy exits the interval $[x_1, x_2]$ is

$$\phi_1(x_2,t) - \phi_1(x_1,t) = \int_{x\in(x_1,x_2)} \frac{\partial\phi_1}{\partial x}$$

assuming the partial derivative $\partial\phi_1/\partial x$ exists. Again, ϕ_1 is assumed unknown.

Finally, the **forcing** or source/sink energy is modeled by a function $Q_1 = Q_1(x,t)$, which is presumed in some fashion given or prescribed, with

$$[Q_1] = \frac{[energy]}{LT}$$

so that the **rate at which energy independently increases in the rod** along the interval $[x_1, x_2]$ is modeled by

$$\int_{x\in(x_1,x_2)} Q_1. \tag{16}$$

Thus, Q_1 may also be viewed as a kind of energy density, but it is actually an **energy density rate** so we have units

$$\left[\int_{x\in(x_1,x_2)} Q_1\right] = \frac{[energy]}{T}.$$

The forcing may be associated with external heating or cooling/extraction of heat energy. The forcing may also be associated with internal (chemical) reactions which may also depend on other quantities. For example, it would not be unusual to have $Q_1 = Q_1(x, t, \theta_1, \phi_1)$, but the dependence on thermal energy density and thermal energy flux would be assumed to be given by some known physical/chemical principle, e.g., at a given thermal energy density θ_1, heat is generated internally at a (density) rate of $Q_1(\theta_1)$.

5.1 Energy Accounting

With these quantities the **accounting of energy transfer** into and out of the portion $[x_1, x_2]$ of the rod is expressed by

$$\frac{d}{dt}\int_{x\in(x_1,x_2)} \theta_1 = -[\phi_1(x_2,t) - \phi_1(x_1,t)] + \int_{x\in(x_1,x_2)} Q_1.$$

In the situation without forcing $Q_1 \equiv 0$, the relation

$$\frac{d}{dt}\int_{x\in(x_1,x_2)} \theta_1 = -[\phi_1(x_2,t) - \phi_1(x_1,t)],$$

is often referred to as an expression of **conservation of energy**. Assuming $\theta, \phi \in C^1[a, b]$, the accounting relation may be written as a single integral:

$$\int_{x\in(x_1,x_2)} \left(\frac{\partial\theta_1}{\partial t} + \frac{\partial\phi_1}{\partial x} - Q_1\right) = \int_{x\in(a,b)} \left(\frac{\partial\theta_1}{\partial t} + \frac{\partial\phi_1}{\partial x} - Q_1\right) \chi_{(x_1,x_2)} = 0.$$

Assuming this relation holds for all subintervals (x_1, x_2), we conclude

$$T_1(x,t) = \frac{\partial\theta_1}{\partial t} + \frac{\partial\phi_1}{\partial x} - Q_1 = 0 \qquad \text{for } (x,t) \in (a,b)\times(0,T)$$

on whatever time interval $(0, T)$ the evolution is defined. We recall that assuming $T(x_0, t_0) > 0$ at some position x_0 and some time t_0, along with minimal regularity, say L^1_{loc} for T_1, leads to the contradiction

$$0 = \int_{x\in(a,b)} T_1(x,t_0)\,\chi_{(x_0-\delta,x_0+\delta)} > 0$$

for $\delta > 0$ small enough. This contradiction implies $T_1 \leq 0$, and the same reasoning under the assumption $T_1(x_0, t_0) < 0$ gives $T_1 \geq 0$. It will be noted that this argument is essentially that used to prove the **fundamental lemma of the calculus of variations**, and the conclusion is essentially the same as well.

The result is a partial differential equation which may be considered a first version of the 1D heat equation:

$$\frac{\partial \theta_1}{\partial t} = -\frac{\partial \phi_1}{\partial x} + Q_1. \tag{17}$$

This is generally considered an equation for the unknown heat density θ_1 and thermal flux field ϕ_1 with a given forcing Q_1.

5.2 Temperature and Fourier's Law

We introduce a new (spatial) quantity, **temperature** $u = u(x,t)$ measured in a new fundamental unit which we represent by $[temp]$. Historically, temperature was difficult to distinguish from thermal energy, but eventually given the ability to measure both separately it appeared that **different substances require different amounts of heat energy to produce the same change in temperature**. As a result, the **law of specific heat** in the form

$$u = \sigma \rho_1 \theta_1 \tag{18}$$

was proposed to model this distinction, where σ is a material constant called the **specific heat capacity** and ρ_1 is the **1D mass density**:

$$[\sigma] = \frac{[temp]}{[energy]M} \qquad \text{and} \qquad [\rho_1] = \frac{M}{L}.$$

Closer examination suggested that improved modeling could be obtained by considering $\sigma = \sigma(u)$ rather than as a constant, though the approximation according to which σ is considered constant for a certain range of temperatures is also widely used, and we will use it below.

Temperature, then, is a scale for determining heat energy density, in a certain sense, **arbitrarily** with respect to a **given substance** and some other physical property (for example spatial expansion) of the given substance in particular. It may be noted that this constitutes a very different kind of fundamental units, and what is being done here is perhaps worth understanding: One takes a specific substance, for example mercury, and a specific **sample mass** of that substance. Then one records the volume of that specific sample mass of mercury at two different times (presumably) in thermal equilibrium but with different **constant total thermal energy**. A sample of a different substance is said to have **the same temperature as the mercury** if there is no thermal energy exchange when the two samples are put in **thermal contact**. Assigning numbers (a certain number of degrees) to the smaller volume, say $0°$, and a different number of degrees, say $n°$, to the larger volume. The intermediate volumes are assigned intermediate degree measure according to **volume ratio**. That is to say, if $0°$ is the temperature of a volume v_0 of mercury and $n°$ is the temperature of a volume v_n of (the same mass of) mercury, then when the same mass of mercury (in thermal equilibrium) has volume v we say that volume v has temperature $k°$ where

$$\frac{v - v_0}{v_n - v_0} = \frac{k}{n}.$$

On the face of it, there is no reason to believe that **the thermal energies** of the three volumes v_0, v, and v_1 of mercury share the same proportions determined by spatial expansion. That is to say, it is not at all obvious that spatial expansion depends linearly on total thermal energy. Nevertheless, in a practical sense, this (at least) seems to give good approximation, so that the thermal energy of a sample of mercury may be assumed to satisfy

$$u = \sigma_{Hg} \rho_{Hg} \theta_{Hg}$$

and the thermal energy of other substances may be assumed to be determined by comparison. Thus, we arrive at the **law of specific heat** (18).

With the introduction of temperature u, it was also observed that heat energy flows (more or less by definition) from regions of higher temperature to regions of lower temperature. Accordingly, Fourier suggested the relation

$$\phi_1 = -K_1 u_x. \tag{19}$$

Actually, the full suggestion of Fourier was applicable to any dimension (and three dimensions in particular) in the form

$$\vec{\phi} = -KDu \tag{20}$$

where $\vec{\phi}$ is the vector valued thermal flux field and Du is the **spatial temperature gradient**, but we are primarily interested in the 1D case here in which the form (19) may be considered **Fourier's law**. The "constant" of proportionality $K = K_1$ is called the **thermal conductivity** and one sees

$$[K_1] = \frac{[energy]L}{[temp]T} \qquad \text{because} \qquad [Du] = \frac{[temp]}{L}.$$

As a physical constant, the units of K_1 are a little strange. The 3D case given in (20) gives units

$$[K] = \frac{[energy]}{[temp]TL}$$

which make some sense as the amount of energy required to increase the difference in the temperature values measured at two points $\mathbf{p}_1$ and $\mathbf{p}_2$ in a (homogeneous) substance by one degree, in one unit of time, if the points $\mathbf{p}_1$ and $\mathbf{p}_2$ are one unit of length distant from each other. In one lower dimension, with

$$[K_2] = \frac{[energy]}{[temp]T}$$

it is already not entirely clear how to interpret the areal (or laminar) conductivity physically. The description above involving two points with a distance of one unit of length between them still seems to make sense, but writing

$$[K_2] = \frac{[energy]L}{[temp]TL}$$

one is faced with the units of $[energy]L$ or alternatively $[force]L^2$. But what is force $\times$ area? If we knew that, then we would seemingly have an interpretation. The same comment applies in the 1D case with force $\times$ volume. Presumably these units, $[force]L^2$ and $[force]L^3$, are natural units in some context, but they do not appear to be ones we think about often.

Finally, it may be noted (as one should expect) that the model suggested by Fourier's law is known to be approximate, in the sense that the model can be improved by assuming a dependence on temperature $K_1 = K_1(u)$. One may also naturally allow, for example, dependence on mass density and a number of other quantities in addition to position. It is not usual to allow a non-autonomous dependence of K_1 on time.

5.3 The Heat Equation

According to the law of specific heat and Fourier's law, the PDE (17) becomes a partial differential equation for the single scalar quantity u:

$$\frac{\partial}{\partial t}(\sigma\rho_1 u) = \frac{\partial}{\partial x}(K_1 u_x) + Q_1. \tag{21}$$

It is usual to assume **no mass flow** in the modeling of heat diffusion/transfer, so that ρ is independent of both explicit and implicit dependence on time. More generally, it is often assumed that σ, ρ_1 and K_1 are all constants, so that (21) takes the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q. \tag{22}$$

where

$$k = \frac{K_1}{\sigma\rho_1} \qquad \text{and} \qquad q = \frac{Q_1}{\sigma\rho_1}.$$

I have left off the subscripts $k = k_1$ and $q = q_1$ here, but the same interpretations usually hold for the equation in this form under the assumption $\sigma\rho_1 = 1$, so that $k = k_1$ is considered a thermal conductivity and q_1 a thermal forcing with

$$[k_1] = \frac{[energy]L}{[temp]T} \qquad \text{because} \qquad [q_1] = \frac{[energy]}{LT}.$$

We have also noted, that under these assumptions it is possible to scale in space setting $u(x,t) = \tilde{u}(\sqrt{k}\,x, t)$ on $(a,b) \times (0,T)$ where $\tilde{u}$ satisfies (22) with $\tilde{u}$ and q defined on $(a\sqrt{k}, b\sqrt{k}) \times (0,T)$ so that

$$u_{xx} = k\tilde{u}_{\xi\xi} = \tilde{u}_t - q = u_t - f$$

where $f : (a,b) \times (0,T) \to \mathbb{R}$ by $f(x,t,u,...) = q(\sqrt{k}\,x, t, \ldots)$. This results in a final simplest form of the heat equation given by

$$u_t = u_{xx} + f \tag{23}$$

where again, the constant $\sigma\rho_1$ is interpreted to have the value $\sigma\rho_1 = 1$ and $k_1 = 1$ is interpreted as unit conductivity with f a heat energy forcing (density rate).

5.4 Interpretation

We are now in a position to make a (hopefully useful) physical interpretation of μ and the Dirac distribution as forcing terms for the equation (23) in relation to the weak and distributional solutions u_δ and $G(x,\xi)$. First of all, we note that (23) reduces to the 1D Poisson equation when $u = u(x)$ and $f = f(x)$ are independent of time. Thus, the boundary value problem for the 1D Poisson equation represents the **steady state** for (23) **with time independent forcing**.

The forcing function μ given in (7) represents a steady input of heat energy uniformly distributed along the interval $[\xi - \delta, \xi + \delta]$ with total rate of one unit of energy per time as represented by the condition

$$\int_0^L \mu(x)\, dx = 1 \tag{24}$$

in accordance with (16). The function u_δ is the **steady state response** to this particular heat input/forcing with the temperature of the ends fixed with $u(0,t) = u(L,t) = 0$. Fourier's law may be used to model the heat energy exiting at the endpoints with total exiting rate given by

$$-[u'(L) - u'(0)] = -[u_x(L,t) - u_x(0,t)] = \frac{\xi}{L} + \frac{L-\xi}{L} = 1. \tag{25}$$

Thus, the total rate of energy input (24) and rate of exiting (25) are the same, as one should expect.

In the limit as $\delta \to 0$, one obtains $u_\delta \to G$ so that the Green's function is the **formal response** to a concentrated unit point source of heat energy (at ξ) of fixed unit magnitude. This is not necessarily a physically realizable system, but that is not really the point. We have already made a formal calculation suggesting

$$u(x) = \int_0^L G(x,\xi) f(\xi)\, d\xi$$

gives a solution to (1) for any forcing function f. We will justify and elaborate this assertion below. Furthermore, this general approach/construction and its general characteristics have relatively wide ranging and important applications in many other contexts, so it's worth understanding it well.

Summary

We've considered the Green's function $G : [0,L] \times [0,L] \to \mathbb{R}$ for the ordinary differential operator $Lu = -u''$ considered classically on $C^2[0,L]$ given by

$$G(x,\xi) = \begin{cases} \dfrac{L-\xi}{L}\, x, & 0 \le x \le \xi \\[2ex] -\dfrac{\xi}{L}\,(x-L), & \xi \le x \le L. \end{cases}$$

This may be easily translated to $G : [a,b] \times [a,b] \to \mathbb{R}$ for any interval $I = [a,b]$:

$$G(x,\xi) = \begin{cases} \dfrac{b-\xi}{b-a}\,(x-a), & a \le x \le \xi \\[2ex] -\dfrac{\xi-a}{b-a}\,(x-b), & \xi \le x \le b. \end{cases} \tag{26}$$

In particular, we've considered the interpretation (both mathematically and physically) of G as a solution of the steady state 1D heat equation $u_t - u_{xx} + t$ with generalized forcing given by a Dirac distribution representing the input of one unit of heat energy per unit time at $x = \xi$. These considerations also translate to an arbitrary interval $I = [a, b]$ without significant change. The distributional formulation of "$-u'' = \delta_\xi$" for example is

$$-\int_{(a,b)} u\phi'' = \phi(\xi) \qquad \text{for all } \phi \in C_c^\infty(a, b), \tag{27}$$

and it is easily checked that $G = G(x, \xi)$ given in (26) satisfies this condition with $G(a, \xi) = G(b, \xi) = 0$.

We have considered a more physically realistic "dispersed" heat source forcing given by

$$\mu(x) = \frac{1}{2\delta}\chi_{[\xi-\delta,\xi+\delta]}$$

with $\delta > 0$ and $[\xi - \delta, \xi + \delta] \subset (0, L)$. For the boundary value problem

$$\begin{cases} -u'' = \mu, & \text{on } (0, L) \\ u(0) = u(L) = 0 \end{cases} \tag{28}$$

we found a weak solution $u_\delta : [0, L] \to \mathbb{R}$ given by

$$u_\delta(x) = \begin{cases} (L-\xi)x/L, & 0 \le x \le \xi - \delta \\ -(x-\xi)^2/(4\delta) + \frac{L-2\xi}{2L}(x-\xi) - \frac{4\xi^2 - 4L\xi + \delta L}{4L}, & \xi - \delta \le x \le \xi + \delta \\ -\xi(x-L)/L, & \xi + \delta \le x \le L \end{cases} \tag{29}$$

with $u_\delta \in W^2(0, L) \cap \sqsubset^2[0, L] \cap C^1[0, L]$. In view of the regularity of u_δ given by (29), the weak formulation of $-u'' = \eta$ may take either of two forms, namely:

$$-\int_{(0,L)} u\phi'' = \int_{(0,L)} \mu\phi \qquad \text{for all } \phi \in C_c^\infty(0, L)$$

or

$$\int_{(0,L)} u'\phi' = \int_{(0,L)} \mu\phi \qquad \text{for all } \phi \in C_c^\infty(0, L).$$

In particular, we note that $u_\delta \in W^{2,p}(0, L) \cap W_0^{2,p}(0, L)$ for every $p \ge 1$ with classical first derivative

$$u'(x) = \begin{cases} (L-\xi)/L, & 0 \le x \le \xi - \delta \\ -(x-\xi)/(2\delta) + (L-2\xi)/(2L), & \xi - \delta \le x \le \xi + \delta \\ -\xi/L, & \xi + \delta \le x \le L \end{cases}$$

and weak second derivative $-\mu$. We recall/note that $W_0^{2,p}$ is the closure of C_c^∞ in $W^{2,p}$ with respect to the $W^{2,p}$ norm.

Exercise 5 *Translate this discussion of u_δ to a general interval $I = [a, b]$.*

It is quite interesting to note (and perhaps somewhat unexpected) that the solution u_δ matches G identically for $|x - \xi| \geq \delta$. On the other hand, a classical solution might well be expected for $|x - \xi| \geq \delta$, and this means an affine solution $u_\delta(x) = mx + b$. Furthermore, Fourier's law applied at the endpoints of the interval with reference to the accounting/conservation of energy, in the sense that for steady state the unit rate of heat energy from the forcing should match the heat energy exiting at the endpoints) narrows the possibilities for the affine slopes m.

We now attempt a somewhat more sophisticated approach giving a classical solution of

$$\begin{cases} -u'' = \mu_\delta & \text{on } (a, b) \\ u(a) = u(b) = 0 \end{cases} \tag{30}$$

with $\mu = \mu_\delta \in C_c^\infty(a, b)$ a specific smooth forcing which constitutes an **approximate identity** (which is an interesting construction in itself).

As a final comment before we embark on this endeavor, one can look for the solution as a C^∞ version of the previous solution u_δ, and one can ask if we obtain a C^∞ modification of G only supported for $|x - \xi| < \delta$ and which is also a classical solution.

6 Fundamental Solution, Convolution and Mollification

6.1 Mollification

Let us begin with consideration of the function $\mu_1 : \mathbb{R} \to \mathbb{R}$ by

$$\mu_1(x) = \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

It can be shown that $\mu_1 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \mu_1 = [-1, 1]$. A plot of μ_1 is shown in Figure 5. It might be assumed that this function is of some particular interest and, working under that assumption, we might be interested in some properties of μ_1 among which are the following:

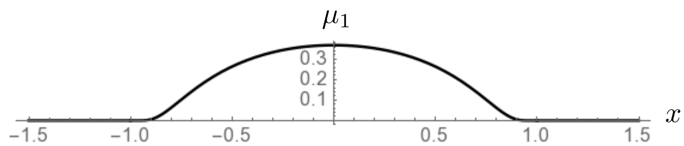

Figure 5: A standard test function μ_1.

1. $\mu \geq 0$ and μ_1 is even: $\mu_1(-x) = \mu_1(x)$.
2. μ_1 is increasing and positive for $-1 < x < 0$ with a unique maximum value $\mu_1(0) = 1/e \doteq 0.36788$.
3. μ_1 is positive and decreasing with $\mu_1'(x) < 0$ for $0 < x < 1$.
4. $\int \mu_1 \doteq 0.44399$.

The function μ_1, or a function with similar regularity and monotonicity properties, is sometimes called a **standard test function**. A **mollifier** or **approximate identity** is obtained from μ_1 as follows: Let $\delta > 0$ and set

$$\mu_\delta(x) = \frac{1}{\delta m}\mu_1(x/\delta) \qquad \text{where} \qquad m = \int \mu_1.$$

Notice, first of all, that $\operatorname{supp}\mu_\delta = [-\delta, \delta]$ and

$$\int \mu_\delta = \frac{1}{\delta m}\int_{-\delta}^{\delta} \mu_1(x/\delta)\, dx = \frac{1}{m}\int_{-1}^{1} \mu_1(\xi)\, d\xi = 1.$$

The **convolution** (integral) of a function $u \in L^1_{loc}(\mathbb{R})$ with μ_δ is called a **mollification** of u:

$$\mu_\delta * u(x) = \int_{\xi \in \mathbb{R}} \mu(\xi)\, u(x - \xi).$$

This function $\mu_\delta * u : \mathbb{R} \to \mathbb{R}$ satisfies $\mu \in C^\infty(\mathbb{R})$. Convolution also interacts nicely with u on intervals in $\mathbb{R}$ where u is constant or affine: If $u(x) \equiv c$ for $x \in (x_0-T, x_0+T)$ with $T > \delta$, then

$$\mu_\delta * u(x) = \int_{\xi \in (-\delta,\delta)} \mu_\delta(\xi)\, u(x - \xi) = c$$

as long as $x-\xi \in [x_0-T, x_0+T]$ for $|\xi| \le \delta$, that is, for $x \in [x_0-(T-\delta), x_0+(T-\delta)]$. More generally, if $u(x) = mx + b$ for $x \in (x_0 - T, x_0 + T)$ with $T > \delta$, then

$$\begin{aligned}\mu_\delta * u(x) &= \int_{\xi\in(-\delta,\delta)} \mu_\delta(\xi)\, u(x-\xi)\\ &= b + m\int_{\xi\in(-\delta,\delta)} \mu_\delta(\xi)\,(x-\xi)\\ &= b + mx - m\int_{\xi\in(-\delta,\delta)} \mu_\delta(\xi)\,\xi\\ &= mx+b,\end{aligned}$$

again, as long as $x \in [x_0 - (T-\delta), x_0 + (T-\delta)]$. Note that the quantity $\int \xi\mu_\delta(\xi) = 0$ because μ_δ is even. Explicitly,

$$\int_{-\delta}^{0} \xi\mu_\delta(\xi)\, d\xi + \int_0^\delta \xi\mu_\delta(\xi)\, d\xi = \int_\delta^0 x\mu_\delta(-x)\, dx + \int_0^\delta \xi\mu_\delta(\xi)\, d\xi = 0.$$

6.2 Fundamental Solution

Even though it is a simple observation we have made previously, it is striking that the values of the derivatives of the Green's function at the endpoints, counted as flux values with the appropriate signs, according to (26) satisfy

$$G_x(a,\xi) - G_x(b,\xi) = \frac{b-\xi}{b-a} + \frac{\xi-a}{b-a} = 1.$$

This means, in particular, that the most symmetric solution, obtained with the source at $\xi = (a+b)/2$ always consists of piecewise affine solutions of $\Delta u = u'' = 0$ with slopes $m = \pm 1/2$. We extend and translate these symmetric forms to obtain an even function $\Phi : \mathbb{R} \to \mathbb{R}$ given by

$$\Phi(x) = -\frac{1}{2}\,|x|$$

which we call the **fundamental solution**. We wish to verify several other properties of the fundamental solution which have analogues in higher dimensions Φ. First of all, given any $f \in C_c^2(\mathbb{R})$ we may set

$$u_0(x) = \Phi * f(x) = \int_{\xi\in\mathbb{R}} \Phi(\xi) f(x-\xi) \qquad (31)$$

to obtain a function $u_0 \in C^2(\mathbb{R})$ satisfying

$$-\Delta u_0 = -u_0'' = f. \tag{32}$$

Consequently, we can (easily) solve

$$\begin{cases} \Delta w = w'' = 0, & \text{on } [a,b] \\ w(a) = u_0(a), \ w(b) = u_0(b) \end{cases} \tag{33}$$

to obtain a function $u(x) = u_0(x) - w(x)$ satisfying the homogeneous boundary value problem

$$\begin{cases} -\Delta u = -u'' = f, & \text{on } [a,b] \\ u(a) = u(b) = 0 \end{cases} \tag{34}$$

for Poisson's equation.

Following a formal simplified version of this construction, we observe that the general Green's function may be expressed as

$$G(x,\xi) = \Phi(x-\xi) - w(x,\xi) \tag{35}$$

where $w = w(x,\xi)$ satisfies

$$\begin{cases} \Delta w = w'' = 0, & \text{on } [a,b] \\ w(a) = \Phi(a-\xi), \ w(b) = \Phi(b-\xi). \end{cases} \tag{36}$$

Finally, we put everything together (in a certain sense) and explicitly solve

$$\begin{cases} -\Delta u(x) = -u''(x) = \mu_\delta(x-\xi), & \text{on } [a,b] \\ u(a) = u(b) = 0 \end{cases} \tag{37}$$

for a function $u = u_\delta \in C_c^\infty[a,b]$.

Since $\Phi \in C^0(\mathbb{R})$ and $f \in C_c^2(\mathbb{R})$ has compact support we may differentiate under the integral sign in (31) to obtain two continuous derivatives

$$u_0'(x) = \Phi * f'(x) = \int_{\xi\in\mathbb{R}} \Phi(\xi) f'(x-\xi)$$

and

$$u_0''(x) = \Phi * f''(x) = \int_{\xi\in\mathbb{R}} \Phi(\xi) f''(x-\xi).$$

Calculating from the last expression, we find

$$
\begin{aligned}
-u_0''(x) = -\Phi * f''(x) &= -\int_{\xi\in\mathbb{R}} \Phi(\xi)\, f''(x-\xi) \\
&= \frac{1}{2}\int_{\xi\in\mathbb{R}} |\xi|\, f''(x-\xi) \\
&= -\frac{1}{2}\int_{-\infty}^{0} \xi\, f''(x-\xi)\, d\xi + \frac{1}{2}\int_{0}^{\infty} \xi\, f''(x-\xi)\, d\xi \\
&= -\frac{1}{2}\left[-\xi f'(x-\xi)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} f'(x-\xi)\, d\xi\right] \\
&\qquad\qquad + \frac{1}{2}\left[-\xi f'(x-\xi)\Big|_{0}^{\infty} + \int_{0}^{\infty} f'(x-\xi)\, d\xi\right] \\
&= -\frac{1}{2}\left[-f(x-\xi)\Big|_{-\infty}^{0}\right] + \frac{1}{2}\left[-f(x-\xi)\Big|_{0}^{\infty}\right] \\
&= \frac{1}{2}f(x) + \frac{1}{2}f(x) = f(x).
\end{aligned}
$$

Assuming $\operatorname{supp} f \subset [a,b]$ and $x \in \mathbb{R}\backslash(a,b)$ we make an additional calculation. First for $x \le a$,

$$
\begin{aligned}
u_0(x) = \Phi * f(x) = \int_{\xi\in\mathbb{R}} \Phi(\xi)\, f(x-\xi) &= -\frac{1}{2}\int_{\xi\in\mathbb{R}} |\xi|\, f(x-\xi) \\
&= \frac{1}{2}\int_{-\infty}^{0} \xi\, f(x-\xi)\, d\xi - \frac{1}{2}\int_{0}^{\infty} \xi\, f(x-\xi)\, d\xi \\
&= -\frac{1}{2}\int_{\infty}^{x} (x-\eta)\, f(\eta)\, d\eta \\
&= \left(\frac{1}{2}\int_{x}^{\infty} f(\eta)\, d\eta\right) x - \frac{1}{2}\int_{x}^{\infty} \eta\, f(\eta)\, d\eta \\
&= \left(\frac{1}{2}\int_{a}^{b} f(\eta)\, d\eta\right) x - \frac{1}{2}\int_{a}^{b} \eta\, f(\eta)\, d\eta \\
&= mx - \beta,
\end{aligned}
$$

where

$$
m = \frac{1}{2}\int_{a}^{b} f(\eta)\, d\eta \qquad \text{and} \qquad \beta = \frac{1}{2}\int_{a}^{b} \eta\, f(\eta)\, d\eta. \tag{38}
$$

Similarly, for $x \geq b$,

$$
\begin{aligned}
u_0(x) = \Phi * f(x) = \int_{\xi\in\mathbb{R}} \Phi(\xi)\, f(x-\xi) &= -\frac{1}{2}\int_{\xi\in\mathbb{R}} |\xi|\, f(x-\xi) \\
&= \frac{1}{2}\int_{-\infty}^{0} \xi\, f(x-\xi)\, d\xi - \frac{1}{2}\int_{0}^{\infty} \xi\, f(x-\xi)\, d\xi \\
&= \frac{1}{2}\int_{x}^{-\infty} (x-\eta)\, f(\eta)\, d\eta \\
&= -\left(\frac{1}{2}\int_{-\infty}^{x} f(\eta)\, d\eta\right) x + \frac{1}{2}\int_{-\infty}^{x} \eta\, f(\eta)\, d\eta \\
&= -\left(\frac{1}{2}\int_{a}^{b} f(\eta)\, d\eta\right) x + \frac{1}{2}\int_{a}^{b} \eta\, f(\eta)\, d\eta \\
&= -mx + \beta.
\end{aligned}
$$

Thus, we have a solution of (32). The fact that $u_0 \in C^2(\mathbb{R})$ along with the calculation of values outside $\operatorname{supp} f$, the support of the forcing, implies a couple things that are sort of amazing. Let's start back with a forcing function $f \in C^2_c(\mathbb{R})$. We have taken a specific such function and plotted it on the left in Figure 6. Rather than simply requiring $\operatorname{supp} f \subset [a, b]$, let us assume more precisely that $a = \min \operatorname{supp} f$ and $b = \max \operatorname{supp} f$. One thing these calculations say is that if you want to construct

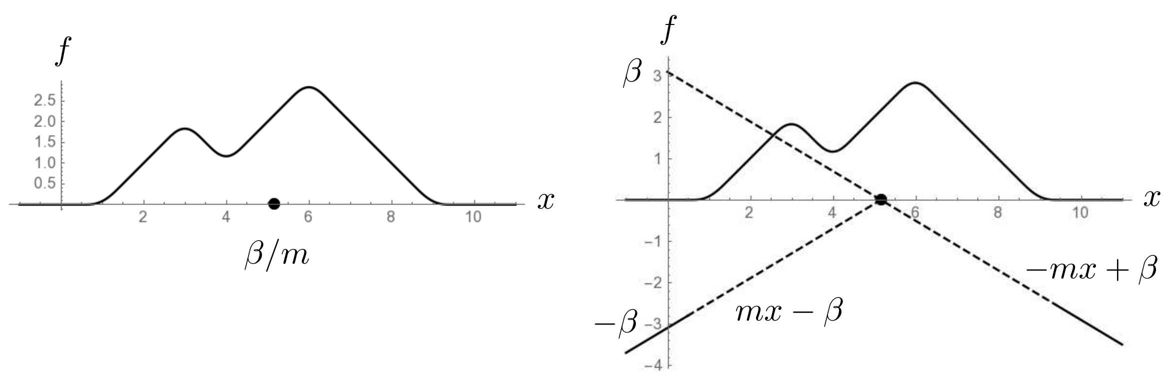

Figure 6: A forcing function $f \in C^2_c(\mathbb{R})$ with the values of u_0 outside $\operatorname{supp} f$. In this case, $\operatorname{supp} f = [0.5, 9.5]$ and the center of mass is located around $x = 31/6 \doteq 5.17$ (slightly to the right of the center of the support interval) and we have actually plotted $u_0/10$ for visualization purposes.

a solution for $-u_0'' = f$, then you can start by computing the "center of mass" of f where the "mass" of the forcing is, as we might expect,

$$2m = \int_a^b f(x)\,dx$$

and the "first moment" of the forcing is

$$2\beta = \int_a^b x\,f(x)\,dx,$$

so that the center of mass is

$$\overline{x} = \frac{\beta}{m} = \frac{1}{2m}\int_a^b x\,f(x)\,dx.$$

Next, take the function

$$u_0(x) = -m|x-\overline{x}| \qquad \text{for } x \in \mathbb{R}\backslash\operatorname{supp} f$$

as indicated on the right in Figure 6, and you have your solution outside the support of f. These observations constitute the first somewhat amazing observation.

To complete the picture, solve the initial value problem

$$\begin{cases} -u_0'' = f \quad \text{for } x \in \operatorname{supp} f \\ u_0(a) = ma - \beta,\ u_0'(a) = m, \end{cases}$$

then (amazingly) when you use this formula across $[a, b]$, at the endpoint $x = b$, you will find $u_0(b) = -mx + \beta$. Explicitly,

$$u_0'(x) = m - \int_a^x f(\xi)\,d\xi \quad \text{and} \quad u_0(x) = mx - \beta - \int_a^x \int_a^t f(\xi)\,d\xi\,dt \quad \text{for } x \in \operatorname{supp} f.$$

Applying Fubini's theorem to the iterated integral, we can write

$$u_0(x) = mx - \beta - \int_a^x \int_\xi^x f(\xi)\,dt\,d\xi = mx - \beta - \int_a^x f(\xi)(x-\xi)\,d\xi$$

which can also be written in the form

$$u_0(x) = \left(m - \int_a^x f(\xi)\,d\xi\right)x - \left(\beta - \int_a^x \xi\,f(\xi)\,d\xi\right).$$

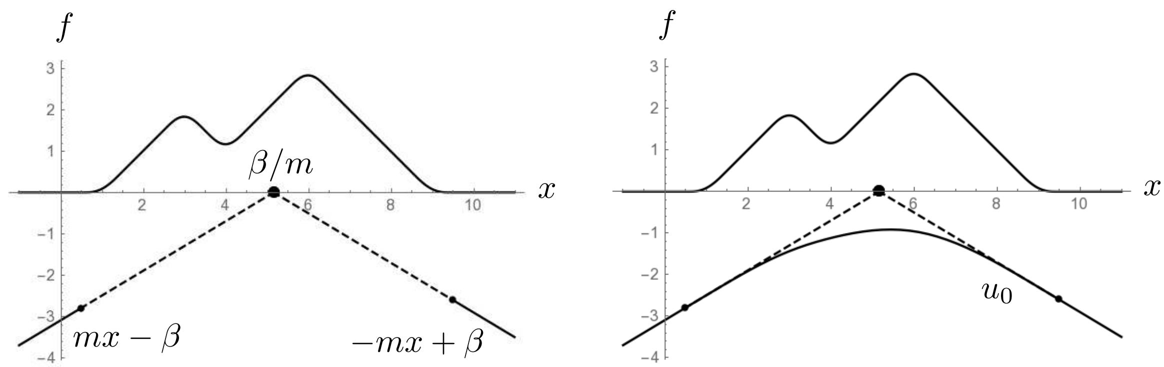

Figure 7: Notice on the left that the points $(a, ma - \beta)$ and $(b, -mb + \beta)$ satisfy $ma - \beta \neq -mb + \beta$. On the right we have filled in, via numerical integration, the the values of u_0 over the interval $\operatorname{supp} f$. (The actual numerical values illustrated are again $u_0/10$.)

There are several directions we can go from here. Perhaps the first thing to notice is that this latter construction, or "recipe," leading to

$$u_0(x) = \begin{cases} mx - \beta, & x \le a \\ \left(m - \int_a^x f(\xi)\, d\xi\right) x - \left(\beta - \int_a^x \xi\, f(\xi)\, d\xi\right), & a \le x \le b \\ -mx + \beta, & x \ge b \end{cases} \tag{39}$$

does not depend on the assumption $f \in C_c^2(\mathbb{R})$, but this formula gives a well-defined C^2 function for any $f \in C_c^0(\mathbb{R})$. It is natural to ask the question

Does this function satisfy $-u_0'' = f$ even for $f \in C_c^0(\mathbb{R})$?

The answer is almost certainly "yes."

Exercise 6 *If $f \in C^1(\mathbb{R})$ then $g = \mu_\delta * f' \in C^\infty(\mathbb{R})$ is a weak derivative of $\mu_\delta * f$. In fact, if $f \in H^1(\mathbb{R})$ with* **weak derivative** *$f' \in L^1_{loc}(\mathbb{R})$, then $g = \mu_\delta * f' \in C^\infty(\mathbb{R})$ is a weak derivative of $\mu\delta * f$.*

Exercise 7 *Try to show $-u_0'' = f$ for $f \in C_c^0(\mathbb{R})$ with u_0 given by the formula above. Hint: Mollify f and apply the formula to $\mu_\delta * f \in C_c^\infty(\mathbb{R})$. Then take a limit as $\delta \to 0$.*

There are other natural questions. For a non-negative forcing function like the one we have used in Figures 6 and 7, it is clear from the mean value theorem for integrals that the apex β/m of the "exterior solution" lies within the support interval of f. This is somewhat less clear if f changes signs.

Exercise 8 *Show $\beta/m \in (a,b)$ where $a = \min\operatorname{supp} f$ and $b = \max\operatorname{supp} f$ in all cases.*

We now turn to the boundary value problem (33). We have used $a = \min\operatorname{supp} f$ and $b = \max\operatorname{supp} f$, but there is no reason to assume the endpoints a and b appearing in (33) are these numbers. Generally, then we may take

$$w(x) = \frac{u_0(b)-u_0(a)}{b-a}x + \frac{bu_0(a)-au_0(b)}{b-a} = \frac{(x-a)u_0(b)+(b-x)u_0(a)}{b-a}. \tag{40}$$

In the situation $a \le \min\operatorname{supp} f$ and $b \ge \max\operatorname{supp} f$, this becomes

$$\begin{aligned} w(x) &= \frac{-mb+\beta-(ma-\beta)}{b-a}x + \frac{b(ma-\beta)-a(-mb+\beta)}{b-a} \\ &= \frac{-m(a+b)+2\beta}{b-a}x + \frac{2mab-(a+b)\beta}{b-a}. \end{aligned} \tag{41}$$

In any case, $u = u_0 - w$ is a solution (and the unique solution) of (34).

Taking $u_0(a) = \Phi(a-\xi) = -(\xi-a)/2$ and $u_0(b) = \Phi(b-\xi) = -(b-\xi)/2$ in (40), we obtain the solution of (36):

$$w(x,\xi) = -\frac{(b-\xi)(x-a)+(\xi-a)(b-x)}{2(b-a)} = -\frac{(a+b)x-2\xi x+(a+b)\xi-2ab}{2(b-a)},$$

and from this we may verify (35):

$$\begin{aligned} \Phi(x-\xi)-w(x,\xi) &= -\frac{1}{2}|x-\xi| + \frac{(b-\xi)(x-a)+(\xi-a)(b-x)}{2(b-a)} \\ &= \begin{cases} (a\xi+bx-x\xi-ab)/(b-a), & a \le x \le \xi \\ (ax+b\xi-x\xi-ab)/(b-a), & \xi \le x \le b \end{cases} \\ &= G(x,\xi). \end{aligned}$$

We have considered the convolution of a test function with an arbitrary function (mollification) and the convolution of the fundamental solution with an arbitrary compactly supported continuous function. Now we combine these considerations and convolve the fundamental solution with $\mu_\delta(x-\xi)$ giving a unit mass forcing centered symmetrically at $x = \xi$. Our calculation leading to (39) above applies directly and we have from (38)

$$m = \frac{1}{2}\int \phi_\delta = \frac{1}{2} \qquad \text{and} \qquad \beta = \frac{1}{2}\int_{\eta\in\mathbb{R}} \eta\phi_\delta(\eta-\xi).$$

Changing variables in the second integral gives

$$\beta = \frac{1}{2}\int_{t\in\mathbb{R}} (t+\xi)\phi_\delta(t) = \frac{\xi}{2}\int_{t\in\mathbb{R}} \phi_\delta(t) = \frac{\xi}{2}.$$

Thus, (39) becomes

$$u_0(x) = \begin{cases} (x-\xi)/2, & x \le \xi-\delta \\ \left(\frac{1}{2} - \int_{\xi-\delta}^x \mu_\delta(t)\,dt\right) x - \left(\frac{\xi}{2} - \int_{\xi-\delta}^x t\,\mu_\delta(t)\,dt\right), & \xi-\delta \le x \le \xi+\delta \\ (\xi-x)/2, & x \ge \xi+\delta \end{cases}$$

which satisfies $u_0 \in C^\infty(\mathbb{R})$ and $-\Delta u_0 = -u_0'' = \mu_\delta$.

Substituting $m = 1/2$ and $\beta = \xi/2$ in (41) with a and b again arbitrary subject to $a < \xi < b$, we find

$$w(x) = \frac{2\xi - (a+b)}{2(b-a)}\,x + \frac{2ab-(a+b)\xi}{2(b-a)}.$$

It is again easily verified that

$$u_\delta(x) = u_0(x) - w(x) \equiv G(x,\xi) \qquad \text{for } x \in [a,b]\backslash(\xi-\delta,\xi+\delta).$$

The C^∞ modification over $(\xi-\delta,\xi+\delta)$ is given by

$$\begin{aligned} u_\delta(x) &= u_0(x) - w(x) \qquad \text{for } |x-\xi| \le \delta \\ &= \frac{b-\xi}{b-a}(x-a) - x\int_{\xi-\delta}^x \mu_\delta(t)\,dt + \int_{\xi-\delta}^x t\,\mu_\delta(t)\,dt \\ &= \left(\frac{b-\xi}{b-a} - \int_{\xi-\delta}^x \mu_\delta(t)\,dt\right) x + \left(\int_{\xi-\delta}^x t\,\mu_\delta(t)\,dt - \frac{b-\xi}{b-a}\,a\right). \end{aligned}$$

Exercise 9 *Notice the function* $u_\delta = u_0 - w \in C^\infty(\mathbb{R})$ *we have just obtained, which is a solution of (37), resembles very strongly*

$$u(x) = (\mu_\delta * G)(x) = \int_{t\in\mathbb{R}} \mu_\delta(t)\, G(t-x,\xi).$$

Is it true that

$$u_\delta(x) = (\mu_\delta * G)(x) = \int_{t\in\mathbb{R}} \mu_\delta(t)\, G(t-x,\xi)?$$

Exercise 10 *Take any solution of (34) based on (41) and (39). Alternatively, take any one of the functions* $u(x) = G(x,\xi)$ *determined by Green's function. Note that the function extends to all of* $\mathbb{R}^1$*. Can you give an interpretation of this solution as a model for the temperature in an infinitely long rod? You may wish to address (and think about) the validity of the law of specific heat for various temperature scales. Are negative temperature scales valid for the heat equation? What happens to the modeling around/with respect to absolute zero?*