Linear Spaces
Some notes for Assignment 2

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1 Spans and sums

For this discussion, let W be a linear space over a field F.

Recall that a **subspace** E of W is a subset $E \subset W$ which is a linear space with respect to the addition and scaling inherited from W.

In order to show a subset $E \subset W$ is a subspace, it is enough to verify the following three conditions:

1. E is nonempty,
2. E is closed under addition which means $v + w \in E$ whenever $v, w \in E$, and
3. E is closed under scaling, i.e., $cv \in E$ whenver $c \in F$ and $v \in E$.

We can say all the additional properties required to make E a subspace are inherited by E from W.

Two important kinds of subspaces are **spans** and **sums**.

1.1 Spans

Given a subset $S \subset W$, the **span** of S denoted $\text{span}(S)$ is the collection of all linear combinations of elements of S. Recall that a **linear combination** is an element of W having the form

$$\sum_{n=1}^{\ell} a_n v_n = a_1 v_1 + a_2 v_2 + \cdots + a_\ell v_\ell$$

where $a_1, a_2, \ldots, a_\ell \in F$ and $v_1, v_2, \ldots, v_\ell \in W$. Thus

$$\text{span}(S) = \left\{ \sum_{n=1}^{\ell} a_n v_n : a_1, a_2, \ldots, a_\ell \in F,\ v_1, v_2, \ldots, v_\ell \in S \right\}.$$

Given any subset $S \subset W$, you can show $\text{span}(S)$ is a subspace.

Given a certain subspace $E \subset W$, we say S is a **spanning set** for E if

$$\text{span}(S) = E.$$

Note that it is always true that E is a spanning set for E. Usually there are spanning sets for a linear space W (or a subspace E) that are smaller than the space itself. This observation leads to the notion of a **minimal spanning set**. A spanning set S is a **minimal spanning set** for W if any spanning set $S_0 \subset S$ for W satisfies $S_0 = S$.

Exercise 1 True or false: If S is a minimal spanning set for W, then $S \subset S_1$ for every other spanning set for W.

If there is at least one spanning set for W with finitely many elements, then the linear space W is said to be **finite dimensional**. Otherwise, W is said to be **infinite dimensional**. It takes more work to properly understand and define the **dimension** of a linear space W, but this is not too much trouble in the situation where W is finite dimensional.

2 Sums

Remember a span is formed by taking linear combinations of elements in a set S. The set S does not have to have finitely many elements.

Another important kind of subspace is a sum of subspaces. Here, we start with a collection of finitely many subspaces $E_1, E_2, \ldots, E_m$ of the given linear space W. You can check that

$$E_1 + E_2 + \cdots + E_m = \left\{ \sum_{n=1}^{m} v_n : v_n \in E_n \text{ for } 1, 2, \ldots, m \right\},$$

that is the collection of sums of elements from W consisting of m terms one from each subspace E_n, is a subspace of W. This is called the **sum** of the subspaces $E_1, E_2, \ldots, E_m$.

Note that $E_1 + E_2 + \cdots + E_m$ contains each subspace E_n for $n = 1, 2, \ldots, m$, and so the sum contains the union

$$\bigcup_{n=1}^{m} E_n.$$

Typically the sum will be strictly bigger than the union.

A sum of subspaces $E_1 + E_2 + \cdots + E_m$ is said to be a **direct sum** if each element

$$v_1 + v_2 + \cdots + v_m \in E_1 + E_2 + \cdots + E_m$$

has a unique representation as a sum with $v_n \in E_n$ for $n = 1, 2, \ldots, m$.

If one adds the subspaces

$$E_1 = \{(0, z_2, z_3) : z_2, z_3 \in \mathbb{C}\}$$

and

$$E_2 = \{(z_1, 0, z_3) : z_1, z_3 \in \mathbb{C}\}$$

in $\mathbb{C}^3$, one gets $E_1 + E_2 = \mathbb{C}^3$. This is not a direct sum because

$$(0,0,0) = (0,0,-i) + (0,0,i) \qquad \text{with} \qquad (0,0,-i) \in E_1 \quad \text{and} \quad (0,0,i) \in E_2,$$

but also

$$(0,0,0) = (0,0,-2) + (0,0,2) \qquad \text{with} \qquad (0,0,-2) \in E_1 \quad \text{and} \quad (0,0,2) \in E_2$$

Thus the choice of the vectors we are adding together is not unique.

Notice that in the example above $E_1 = \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_2 = (0,1,0) \in \mathbb{C}^3$ and $\mathbf{e}_3 = (0,0,1) \in \mathbb{C}^3$.

On the other hand, if $E_3 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}$ and $E_4 = \{\alpha(1, i, 1+i) : \alpha \in \mathbb{C}\} = \operatorname{span}\{(1, i, 1+i)\}$, then

$$E_3 + E_4 = \mathbb{C}^3$$

is a direct sum. To see this we can use some linear combinations: First of all, given $(z_1, z_2, z_3) \in \mathbb{C}^3$

$$(z_1, z_2, z_3) = \left(z_1 - \frac{z_3}{2}(1-i), z_2 - \frac{z_3}{2}(1+i), 0\right) + \frac{z_3}{2}(1-i)(1, i, 1+i)$$

with

$$\left(z_1 - \frac{z_3}{2}(1-i), z_2 - \frac{z_3}{2}(1+i), 0\right) \in E_3 \qquad \text{and} \qquad \frac{z_3}{2}(1-i)(1, i, 1+i) \in E_4.$$

This means $\mathbb{C}^3 = E_3 + E_4$.

On the other hand, if

$$(z_1, z_2, 0) + \alpha(1, i, 1 + i) = (w_1, w_2, w_3) \in \mathbb{C}^3,$$

then z_1, z_2 and α must satisfy

$$\begin{cases} z_1 + \alpha = w_1 \\ z_2 + \alpha i = w_2 \\ \alpha(1 + i) = w_3. \end{cases}$$

By the last equation, there is only one possible choice for α. That is,

$$\alpha = \frac{w_3}{1 + i} = \frac{w_3}{2}(1 - i).$$

Then there are also unique choices for z_1 and z_2:

$$z_1 = w_1 - \frac{w_3}{2}(1 - i) \qquad \text{and} \qquad z_2 = w_2 - \frac{w_3}{2}(1 + i).$$

Thus, the repesentation in $E_3 + E_4$ is unique, and $E_3 + E_4$ is a direct sum. In this case we write

$$E_3 + E_4 = E_3 \oplus E_4$$

or

$$E_3 + E_4 = E_3 \bigoplus E_4$$

if we want to make the symbol $\oplus$ big.

In general, if $E_1 + E_2 + \cdots + E_m$ is a direct sum, then we write

$$E_1 + E_2 + \cdots + E_m = \bigoplus_{n=1}^{m} E_n.$$

Direct sums have a relation to minimal spanning sets (in finite dimensional linear spaces) and to another notion with which you may be familiar.

Definition 1 A set $S \subset W$ is said to be a **linearly independent set** if whenever

$$\sum_{n=1}^{k} a_n v_n = \mathbf{0}$$

is a linear combination in $\text{span}(S)$ equal to the zero vector, then there must hold

$$a_1, a_2, \ldots, a_k = 0.$$

A set S is **linearly dependent** if there exists a linear combination

$$\sum_{n=1}^{k} a_n v_n = \mathbf{0}$$

with $a_1, a_2, \ldots, a_k \in F$ and $v_1, v_2, \ldots, v_n \in S$ but $(a_1, a_2, \ldots, a_k) \neq (0, 0, \ldots, 0) \in F^k$.

Note there is no requirement that the set S have finitely many elements in the definition of linear independence/dependence.

Exercise 2 Is the empty set linearly dependent or linearly independent? Can you tell?

In the special case in which $\#(S) < \infty$, we have a nice relation:

Exercise 3 Show $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if

$$\text{span}\{v_1, v_2, \ldots, v_k\} = \bigoplus_{n=1}^{k} \text{span}\{v_n\}.$$

A linearly independent spanning set is called a **basis**. The countable collection

$$\{1, X, X^2, X^3, \ldots\}$$

is a basis for the linear space of polynomials $F[X]$ over the field F.

A linear space is said to be **finite dimensional** or **finitely generated** if there exists a spanning set with finitely many elements.

Here are some interesting questions:

1. Does every finite dimensional linear space have a basis?

2. Does every linear space have a basis?

3. Does every basis for a finite dimensional linear space have finitely many elements?

4. Does every basis consisting of finitely many elements have the same number of elements?

5. Is every subset of a linearly independent set linearly independent?

6. Does every spanning set for W contain a basis (at least when W is finite dimensional and has a basis)?

7. Is every linearly independent set in W a subset of a basis (at least when W is finite dimensional and has a basis)?

Here is a key to a lot of these questions:

Theorem 1 In a finite dimensional linear space, no linearly independent set has more elements than any spanning set. That is, if S_I is a linearly independent set in W and $\text{span}(S) = W$, then

$$\#(S_1) \leq \#(S).$$

Proof: Notice this result is asserting something about cardinalities. For this reason if I want to give a reasonable proof, I need to be a little careful about counting. One way to do this is to consider finite sets as finite ordered sets. In particular, since W is finitely generated, I know there is a spanning set $S_0 = \{w_1, w_2, \ldots, w_m\}$ with finitely many elements: $\#(S_0) < \infty$. For spanning sets, however, there is no restriction on repetition; it can very well be the case that $w_7 = w_2$. Under normal counting, then, as set like

$$\{1, 2, 3, 4, 5, 6, 2, 7\}$$

should have $\#\{1, 2, 3, 4, 5, 6, 2, 7\} = 7$. I could throw out repeats, but that gets a little complicated at a couple points, so instead I'll just count such a set as ordered and having 8 elements.

Take any initial v_1 in the linearly independent set S_I, then there are scalars $a_1, a_2, \ldots, a_m$ for which

$$v_1 = \sum_{n=1}^{m} a_n w_n.$$

This means there is a linear combination of elements in

$$\{v_1, w_1, w_2, \ldots, w_m\}$$

with

$$1v_1 - \sum_{n=1}^{m} a_n w_n = \mathbf{0}.$$

Notice that not all the coefficients $a_1, a_2, \ldots, a_n$ are all zero here. How do we know? If all these coefficients were zero, then we would get $1v_1 = 1v_1 + 0v_2 + 0v_3 + \cdots = \mathbf{0}$. This contradicts the assumption that S_I is a linearly independent set. Say $a_\ell \neq 0$. Then

$$w_\ell = \frac{1}{a_\ell} v_1 - \sum_{n \neq \ell} \frac{a_n}{a_\ell} w_n$$

where the sum is taken over all indices $n = 1, 2, \ldots, n$ except $n = \ell$. This means $\text{span}(S_1) = W$ where

$$S_1 = \{v_1\} \cup \{w_n : n \neq \ell\}.$$

What is $\#(S_1)$? Again, assuming each of the (ordered) elements $w_1, w_2, \ldots, w_m$ in the initial spanning set are all counted (even if some of them are the same) so that $\#(S_0) = m$, then we have thrown out w_ℓ and added in v_1. Again v_1 may be the same as some remaining element in $\{w_n : n \neq \ell\}$, but I'll count this again as an ordered set with v_1 as the first element, so

$$\#(S_1) = \#(S_0).$$

I hope this makes sense.

In contrast, I'm not assuming the linearly independent set S_I has finitely many elements. I want to prove that. But if I consider the set $S_I \backslash \{v_1\}$ I'll assume v_1 has really been removed, so there is no other "copy" of v_1 hanging around in there. Of course, I'm not really counting the elements in S_I generally. I've only counted one of them...so far.

Of course, one possibility at this point is that there is only one element in the linearly independent set S_I, namely v_1. Then clearly, we have $\#(S_I) = 1 \leq \#(S_1)$.

One thing that is not a possibility is that $S_I \backslash \{v_1\}$ is nonempty but the ordered set $S_1 = \{v_1\} \cup \{w_n : n \neq \ell\}$ has no more elements from S_0 in the ordering. Were this the case, then $S_1 = \{v_1\}$, and remember we still know S_1 spans all of W. Therefore, we would be able to find some $v_2 \in S_I \backslash \{v_1\}$ with $v_2 = cv_1$. This can't be the case since $\{v_1, v_2\}$ is a linearly independent set.

The remaining alternative is that the linearly independent set S_I has not been exhausted, that is, $S_I \backslash \{v_1\} \neq \phi$ and $S_1 \backslash \{v_1\}$ still contains some elements from S_0. In this case, since $\mathrm{span}(S_1) = W$, we can take some element $v_2 \in S_I \backslash \{v_1\}$ and we have

$$v_2 = c_1 v_1 + \sum_{n \neq \ell} a_n w_n.$$

Again, if the coefficients $\{a_n : n \neq \ell\}$ are all zero, then we get $v_2 = cv_1$ or $-cv_1 + v_2 = \mathbf{0}$ which is a contradiction. Thus, there is some w_ν with $\nu \neq \ell$ for which

$$w_\nu = \frac{1}{a_\nu} v_2 - \frac{c_1}{a_\nu} v_1 - \sum_{n \neq \ell, \nu} a_n w_n.$$

Consequently, we can consider the sets

$$S_I \backslash \{v_1, v_2\} \qquad \text{and} \qquad S_2 = (S_1 \cup \{v_2\}) \backslash \{w_\nu\}.$$

The first set has one fewer element, and the second set has the same number of elements but still spans:

$$\mathrm{span}(S_2) = W \qquad \text{and} \qquad \#(S_2) = \#(S_1) = \#(S_0).$$

At this point finding new symbols (like ℓ and ν) for the indices of removed elements from S_0 becomes something of a hassle. But hopefully you can see the only way this can end is if there are no more elements to remove from the linearly independent set S_I, and we have counted off some number of them: $v_1, v_2, \ldots, v_r$ with $r \leq m = \#(S_0) < \infty$.

We have not quite established that $\#(S_I) \leq \#(S)$ for any spanning set S, but we have established that $\#(S_I) \leq \#(S_0)$ for some particular spanning set with finitely many elements, and in particular $\#(S_I) < \infty$.

Now we can repeat the same procedure with any spanning set S. We won't have to count off the elements of S_I, but we can assume $S_I = \{v_1, v_2, \ldots, v_r\}$. We will have to count off some elements of S, at least enough of them to replace with elements from S_I.