ODE

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November 14, 2020

An ordinary differential equation (ODE) is an equation specifying the **derivative** of an unknown function of one real variable. The objective is to determine which function(s) if any, has that specified derivative. Roughly speaking, then, these equations have the form

$$y' = f.$$

The "form" of this equation should be read as being very open to interpretation and rather broad in application. Only the domain of the function y is specified. The co-domain of y, as well as the domain and co-domain of the specification function f, may be, roughly speaking, "anything that makes sense." We will look at various specific examples presently. It should be noted, however, that the defining characteristic here, if there is one, is that the domain of y must be **an interval in the real line** and the derivative of y appearing on the left must be a derivative with respect to the one real variable in this interval. This is what makes an ODE an ODE. Thus, it is natural to begin the subject with a discussion of derivatives with respect to one real variable. In fact, whenever you encounter an ODE, as soon as you know the co-domain of the unknown function y, it is a good idea to ask yourself

What **kind** *of function is the derivative of y?*

If the meaning of this question is not clear to you at the moment, we should endeavor to make it clear.

Exercise 1 *If* $y : \mathbb{R} \to \mathbb{R}$ *by* $y(x) = x^2$, *what are the domain and co-domain of* y'*? Draw pictures of the* **graphs** *of* y *and* y'. *Draw any other useful illustrations that help you understand these functions.*

Exercise 2 *If* $y : \mathbb{R} \to \mathbb{R}^2$ *by* $y(t) = (\cos(2t), \sin(2t))$, *what are the domain and co-domain of* y*? Draw pictures of the* **images** *of* y *and* y'. *Draw any other useful illustrations that help you understand these functions.*

1 FTC equations

The simplest ODEs are equations of the form

$$y' = f(x)$$

where x is the independent/domain variable associated with y and both f and y are real valued. The derivative of y in this case is usually defined as

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}. \tag{1}$$

The expression

$$\frac{y(x+h) - y(x)}{h}$$

is called a **difference quotient**, and for the limit in (1) to make sense the function y needs to be defined for the values $x + h$ with h in some small interval extending both the left and right about zero. A well-known interval of this sort is one with the form

$$B_\epsilon(0) = (-\epsilon, \epsilon) = \{h \in \mathbb{R} : -\epsilon < h < \epsilon\}.$$

Here it is assumed that ϵ is a positive number, and this set is called the **open ball of radius ϵ about zero** or the symmetric **open** interval of length 2ϵ about zero. More generally, the **open ball** of radius $r > 0$ with center $x_0 \in \mathbb{R}$ is

$$B_r(x_0) = \{x \in \mathbb{R} : |x - x_0| < r\}.$$

This kind of set is used to define **open sets of real numbers** in general. To be precise, a set $U \subset \mathbb{R}$ is said to be **open** if for any $x \in U$, there exists some $r > 0$ for which $B_r(x) \subset U$.

Exercise 3 *Show that an open ball is an open set. Find an open set which is not an open ball.*

Exercise 4 *Show that any union of open sets is open.*

Every open interval has the form

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

where a and b are **extended real numbers** with $a < b$. This means a might be $-\infty$ and b might be $+\infty$. Such an interval is the natural domain, from calculus, for a differentiable function of one real variable, and given $y : (a,b) \to \mathbb{R}$ and $x \in (a,b)$, the **derivative** of y, if it exists, is defined by (1). The value of $y'(x)$, if it exists, gives the **slope of the tangent line to the graph** of y at the point $(x, y(x))$. (You should draw pictures of this tangent line and the secant lines whose slopes are given by the difference quotient, so you thoroughly understand how this definition of the derivative works. The derivative also gives the **instantaneous rate at which the values of the function y change with respect to change in the indepdendent variable x.**

Exercise 5 *Given two values x_1 and x_2 of the independent variable x with $x_1 < x_2$, the difference $x_2 - x_1$ is called the* **increment** *of x determined by these values. The increment of y determined by x_1 and x_2 is $y(x_2) - y(x_1)$. Find the* **average rate of change** *of y over the interval $[x_1, x_2]$. Think: The average rate of change is the total "distance" divided by the total "time."*

Beyond elementary calculus there are other intervals that can be of interest. In particular, one is sometimes interested in a real valued function on an interval $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ with $a \in \mathbb{R}$. In such a case, the **one-sided derivative**

$$\lim_{h \to 0^+} \frac{y(a+h) - y(a)}{h}$$

may be of interest. This is a **derivative from the right** or a **right derivative**. There is also a corresponding notion of **left derivative**. One sided derivatives are also sometimes considered in elementary calculus.

Exercise 6 *Show that $y : (a,b) \to \mathbb{R}$ is differentiable at $x \in (a,b)$ if and only if y has both left and right derivatives at x.*

1.1 Important Sets of Functions

The collection of real valued functions which are differentiable at each point in an open interval (a,b), denoted[1] by $\mathrm{Diff}(a,b)$ or $\mathcal{D}^1(a,b)$, is of primary interest in calculus. The subset of these functions which are merely **continuous** is also important, and we denote these functions by $C^0(a,b)$.

[1]The names for these collections of functions are intended to both compress a significant amount of information in a small and convenient "package" on the printed page and to be suggestive of the concept denoted. Thus, when one sees $\mathrm{Diff}(a,b)$ it may be helpful to think *the* **differentiable** *functions on the interval* (a,b), and similalry when one sees $\mathcal{D}^1(a,b)$, one may think *the functions with* **one derivative**.

Definition 1 *A real valued function* $u : A \to \mathbb{R}$ *where* $A \subset \mathbb{R}$ *is any set of real numbers is* **continuous at** $x_0 \in A$ *if for any* $\epsilon > 0$*, there is some* $\delta > 0$ *for which the following holds:*

$$|u(x) - u(x_0)| < \epsilon \qquad \text{whenever } x \in A \text{ and } |x - x_0| < \epsilon.$$

Note that the condition on x *can also be written as* $x \in A \cap B_\epsilon(x)$*.*

A function is **continuous on** A *if it is continuous at every point in* A*.*

the collection of all functions $f : A \to \mathbb{R}$ *which are continuous on* A *is denoted by* $C^0(A)$*.*

Exercise 7 *Show* $\mathrm{Diff}(a, b) \subset C^0(a, b)$ *but* $C^0(a, b) \not\subset \mathrm{Diff}(a, b)$*. Show that if* $u : [a, b] \to \mathbb{R}$ *has a right derivative at* $x = a$*, then* u *is continuous at* a*.*

When we say the limit of the difference quotient in (1) **exists**, we mean (precisely) that there is a real number L for which given any $\epsilon > 0$, there is some $\delta > 0$ such that the following holds:

$$\left| \frac{y(x+h) - y(x)}{h} - L \right| < \epsilon \qquad \text{whenever } |h| < \delta.$$

When such a real number L exists, we call it the **derivative of** y **at** x and write $L = y'(x)$.

Exercise 8 *Show that if* $u : [a, b] \to \mathbb{R}$ *satisfies the following:*

1. $u \in \mathcal{D}^1(a, b)$*, by which we mean the* **restriction** *of* u *to the interval* (a, b)*, denoted by*

$$u\Big|_{(a,b)} : (a, b) \to \mathbb{R},$$

is in $\mathcal{D}^1(a, b)$ *or equivalently that* u *is differentiable at each point in the open interval* (a, b)*,*

2. *u has a right derivative at a, and*

3. *u has a left derivative at b,*

then there is an **extension** $\overline{u} : \mathbb{R} \to \mathbb{R}$ *with* $\overline{u} \in \mathcal{D}^1(\mathbb{R})$ *and*

$$\overline{u}\Big|_{[a,b]} = u.$$

If we write $\mathrm{Diff}[a,b]$ or $\mathcal{D}^1[a,b]$ we can mean either the collection of functions that are differentiable on the open interval (a,b) and have left and right derivatives at b and a respectively, or we can mean the the collection of functions u which have an extension $\overline{u} : (a-\epsilon, b+\epsilon) \to \mathbb{R}$ for some $\epsilon > 0$ with $\overline{u} \in \mathcal{D}^1(a-\epsilon, b+\epsilon)$. These are the same thing.

A third collection of functions appearing in the statements of many theorems from elementary calculus is the collection of functions with a continuous derivative. These are also called the **continuously differentiable** functions and are denoted by $C^1(a,b)$. That is,

$$C^1(a,b) = \{u \in \mathrm{Diff}(a,b) : u' \in C^0(a,b)\}.$$

We will mostly stick to open intervals as the domain of the functions of interest in ODE. It does not hurt, however, to know about other possibilities.

Exercise 9 *If U is any open subset of $\mathbb{R}$, explain why it makes good sense to think about the collection $C^1(U)$ of functions which are continuously differentiable on U.*

If we write $u \in C^1(A)$, we mean there is an open set U with $A \subset U$ and an extension $\overline{u} : U \to \mathbb{R}$ with

$$\overline{u}\big|_A = u.$$

Exercise 10 *Find a function $u \in C^0(a,b)$ with left and right derivatives at b and a respectively and for which*

$$u\big|_{(a,b)} \in C^1(a,b),$$

but $u \notin C^1[a,b]$. Consequently, $C^1[a,b] \subsetneq \mathcal{D}^1[a,b]$.

1.2 The Fundamental Theorem of Calculus

The collections of functions $C^0(a,b)$, $C^1(a,b)$, and $C^1[a,b]$ are often used in simple statements of the fundamental theorem of calculus:

Theorem 1 *(FTC definite integral version) If $f \in C^1[a,b]$, then*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Theorem 2 *(FTC indefinite integral version) If $f \in C^0(a,b)$, then for any $x_0 \in (a,b)$, the function $F : (a,b) \to \mathbb{R}$ by*

$$F(x) = \int_{x_0}^x f(t)\,dt$$

satisfies $F \in C^1(a,b)$ and $F'(x) = f(x)$.

Notice that the fundamental theorem for definite integrals says (roughly) that if you differentiate f first and then integrate the result, you get back the values of f. The indefinite integral version says more: It says that if you integrate first, then you get something you can differentiate, and it tells you what you get for the derivative.

The requirement $f \in C^1[a, b]$ in the definite integral version is natural because the most well-known condition under which the **Riemann integral** of a function exists is that it be continuous:

Theorem 3 *(existence theorem for Riemann integrals) If $u \in C^0[a, b]$, then there exists a number L such that for any $\epsilon > 0$, there is a number $\delta > 0$ for which the following holds:*

Whenever there are points

$$a = x_0 < x_1 < \ldots < x_n = b \tag{2}$$

with $x_j - x_{j-1} < \delta$ for $j = 1, 2, \ldots, n$ and points

$$x_0 \le x_1^* \le x_1 \le x_2^* \le x_2 \le \ldots \le x_{n-1} \le x_n^* \le x_n, \tag{3}$$

then

$$\left| \sum_{j=1}^{n} u(x_j^*)(x_j - x_{j-1}) - L \right| < \epsilon.$$

Under these conditions the limit L is called the **Riemann integral** *of u and we write*

$$L = \int_a^b u(x)\, dx.$$

The construction of the Riemann integral given in this theorem does not require $u \in C^0[a, b]$, but this is a very convenient condition under which the limit defining the Riemann integral is well-defined. The points $x_0 < x_1 < \cdots < x_n$ in (2) are called **partition points** and considered as a set

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$$

they are called a **partition** of the interval $[a, b]$. The points $x_1^*, x_2^*, \ldots, x_n^*$ appearing in (3) are called **evaluation points**. The length of the largest increment in a partition is called **norm of the partitition**:

$$\|\mathcal{P}\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Thus, the condition for the existence of the Riemann integral of any function $u : [a, b] \to \mathbb{R}$ may be expressed as

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} u(x_j^*)(x_j - x_{j-1}) \qquad \text{exists,}$$

and in this case we can write

$$\int_a^b u(x)\, dx = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} u(x_j^*)(x_j - x_{j-1}).$$

Given a partition and evaluation points, the number

$$\sum_{j=1}^{n} u(x_j^*)(x_j - x_{j-1})$$

is called a **Riemann sum**.

Exercise 11 *Draw pictures illustrating the geometric meaning of the Riemann sum and the integral in terms of areas.*

Sometimes a more general version of the fundamental theorem for definite integrals is given which is also somewhat more complicated to state. In particular, it involves another important collection of functions we haven't mentioned. Let $\text{Riem}[a, b]$ denote the collection of all **Riemann integrable functions** on the interval $[a, b]$. The Riemann integrability theorem says $C^0[a, b] \subset \text{Riem}[a, b]$.

Exercise 12 *Find a function* $u \in \text{Riem}[a, b]\backslash C^0[a, b]$.

Exercise 13 *Show the functions* $u_0 : [0, 1] \to \mathbb{R}$ *and* $u_1 : [0, 1] \to \mathbb{R}$ *by*

$$u_0(x) = \begin{cases} 0, & x \in [0, 1]\backslash\mathbb{Q} \\ 1, & x \in [0, 1] \cap \mathbb{Q} \end{cases}$$

and

$$u_1(x) = \begin{cases} 1, & x \in [0, 1]\backslash\mathbb{Q} \\ 0, & x \in [0, 1] \cap \mathbb{Q} \end{cases},$$

where

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \text{ and } p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \right\}$$

denotes the rational numbers as usual, satisfy $u_j \notin \text{Riem}[a, b]$ *for* $j = 0, 1$.

The more general version of the fundamental theorem for definite integrals is the following:

Theorem 4 *If* $f \in \mathrm{Diff}[a,b]$ *and* $f' \in \mathrm{Riem}[a,b]$*, then*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Exercise 14 *Find a function which is differentiable on* $[a,b]$ *and has a Riemann integrable derivative, but is not in* $C^1[a,b]$*.*

Exercise 15 *Prove Theorem 1 using Theorem 2. Does your proof work for Theorem 4?*

Most proofs of Theorems 1, 2, and 4 usually depend on some version of the mean value theorem which, in turn, is usually proved using Rolle's theorem. I will state these theorems with a couple comments below, but first let me give a short proof of Theorem 2:

The difference quotient $[F(x+h) - F(x)]/h$ may be written as

$$\frac{1}{h}\left[\int_{x_0}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right] = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

We wish to show

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x) = \frac{1}{h}\int_x^{x+h} f(x)\,dt$$

or equivalently

$$\lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt\right| = \lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} [f(t) - f(x)]\,dt\right| = 0.$$

Let $\epsilon > 0$. By continuity (of f at x), there is some $\delta > 0$ for which $|f(t) - f(x)| < \epsilon/2$ whenever $|t - x| < \delta$. Taking $|h| < \delta$, all the arguments t appearing in the integrand of

$$\int_x^{x+h} [f(t) - f(x)]\,dt,$$

since they are between x and $x+h$, satisfy $|t - x| \le |h| < \delta$. Consequently, for $|h| < \delta$ we have

$$\left|\frac{1}{h}\int_x^{x+h}[f(t) - f(x)]\,dt\right| \le \frac{1}{|h|}\left|\int_x^{x+h} |f(t) - f(x)|\,dt\right| \le \frac{1}{|h|}\left|\int_x^{x+h}\frac{\epsilon}{2}\,dt\right| = \frac{\epsilon}{2} < \epsilon.$$

This is exactly what it means to have

$$\lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h}[f(t) - f(x)]\,dt\right| = 0. \qquad \square$$

1.3 Other Integrable Functions

You may have noted that $C^1[a,b] \subset C^0[a,b] \subset \text{Riem}[a,b]$. We say these collections of functions have decreasing levels of **regularity**. Of course, our regularity heirarchy can be extended to functions with higher regularity:

$$C^\omega[a,b] \subset C^\infty[a,b] \subset C^k[a,b] \subset \text{Riem}[a,b].$$

As you might guess, a function in $C^\omega[a,b]$ is one with an extension having a power series representation in some open ball with center at each point in $[a,b]$. Similarly, $u \in C^\infty[a,b]$ if u has an extension $\overline{u} \in C^\infty(a-\epsilon, a+\epsilon)$ for some $\epsilon > 0$. Usually extensions, if they exist, are not unique.

Exercise 16 *Show that the extension of a real analytic function in $C^\omega[a,b]$ is unique. This is called* **analytic continuation**.

The open set, and the open interval in particular, is the simplest kind of set on which to consider regularity. We did not, however, define $\text{Riem}(a,b)$ for an open interval. Here is one way to do that: We first set

$$\underset{loc}{\text{Riem}}(a,b) = \bigcap_{0<\epsilon<(b-a)/2} \text{Riem}[a+\epsilon, b-\epsilon].$$

That is, u is **locally Riemann integrable** on (a,b) if (the appropriate restriction of) u is Riemann integrable on the closed subintervals $[a+\epsilon, b-\epsilon]$ for all small enough positive ϵ. It should be noted that according to our definition of $\text{Riem}_{loc}(a,b)$, it may be the case that $u \in \text{Riem}_{loc}(a,b)$ but still

$$\int_a^b u(x)\,dx$$

may not be well-defined.

Exercise 17 *Find a function $u \in \text{Riem}_{loc}(a,b)$ for which*

$$\int_a^b u(x)\,dx$$

is not a well-defined real number. Hint: It is easy to arrange to have

$$\lim_{\epsilon \searrow 0} \int_{a+\epsilon}^{b-\epsilon} f(x)\,dx = +\infty.$$

Here $\lim_{\epsilon\searrow 0}$ *means the same as* $\lim_{\epsilon\to 0^+}$.

Exercise 18 *Find a function* $u \in \text{Riem}_{loc}(a,b)$ *for which*

$$\int_a^b u(x)\,dx$$

is not a well-defined **extended real number**. *Maybe your example from the previous exercise already works.*

Usually, we want functions in $\text{Riem}(a,b)$ to have a finite valued Riemann integral given in the following sense:

$$\text{Riem}(a,b) = \left\{ u \in \underset{loc}{\text{Riem}}(a,b) : \lim_{\epsilon \searrow 0} \int_{a+\epsilon}^{b-\epsilon} u(x)\,dx \text{ exists (in } \mathbb{R}) \right\}.$$

the value of

$$\int_a^b u(x)\,dx$$

for $u \in \text{Riem}(a,b)$ is **defined to be**

$$\int_a^b u(x)\,dx = \lim_{\epsilon \searrow 0} \int_{a+\epsilon}^{b-\epsilon} u(x)\,dx$$

and such an integral is said to be an **improper integral**.

Exercise 19 *Find a function in* $\text{Riem}(a,b) \setminus \text{Riem}[a,b]$.

Exercise 20 *Our discussion of* $\text{Riem}(a,b)$ *only applies when* $a, b \in \mathbb{R}$. *Extend the definition of improper integrals and* $\text{Riem}(a,b)$ *to the case where a and b might have appropriate extended real values.*

Returning to our regularity heirarchy

$$C^\omega[a,b] \subset C^\infty[a,b] \subset C^k[a,b] \subset \text{Riem}[a,b],$$

we note that the functions u_0 and u_1 defined in Exercise 13 are not in $\text{Riem}[a,b]$. Nevertheless, the values of the integrals

$$\int_{(0,1)} u_0 \qquad \text{and} \qquad \int_{(0,1)} u_1$$

are almost obvious.

Exercise 21 *The rational numbers* $\mathbb{Q}$ *are in one to one correspondence with the integers. Such sets are said to be* **countable** *and the rational numbers are one example. Find a bijective function* $\gamma : \mathbb{N} \to [0,1] \cap \mathbb{Q}$. *When you have such a function you can write* $r_j = \gamma(j)$ *as the* j*-th rational number in* $[0,1]$. *So we have*

$$\mathbb{Q} \cap [0,1] = \{r_1, r_2, r_3, \ldots\}$$

where r_1 *is the first rational number in* $[0,1] \cap \mathbb{Q}$ *and* r_2 *is the second one and so on. Now, let* ϵ *be any positive number. Consider the interval*

$$B_{\epsilon_j}(r_j) = (r_j - \epsilon/2^{j+1}, r_j + \epsilon/2^{j+1})$$

centered at r_j. *Calculate the total length of the union of these intervals.*

Note that whatever the total length ℓ *of these intervals is one should have*

$$0 \leq \int_{(0,1)} u_0 \leq \ell.$$

Why?

Determine the values of $\int_{(0,1)} u_0$ *and* $\int_{(0,1)} u_1$. *Hint: Note that* $u_1(x) = 1 - u_0(x)$. *Use the fact that (at least it should be the case that)*

$$\int_{(0,1)} u_1 = \int_{(0,1)} 1 - \int_{(0,1)} u_0$$

where 1 *here in the integrand represents the function with constant value* 1.

I hope you have convinced yourself that u_0 and u_1 are integrable functions which are not in Riem(0, 1) or Riem[0, 1] or any collection of Riemann integrable functions. You might ask then: What kind of functions are they? The answer is that they are **Lebesgue integrable** functions. The collection of Lebesgue integrable functions on an interval, usually taken to be an open interval, is $L^1(a,b)$. These functions provide a natural extension of our regularity heirarchy to less regular functions:

$$\text{Riem}[a,b] \subset L^1(a,b) \qquad \text{and} \qquad \text{Riem}(a,b) \subset L^1(a,b).$$

Exercise 22 *Write down the natural definition of* $L^1_{loc}(a,b)$ *and show* $\text{Riem}_{loc}(a,b) \subset L^1_{loc}(a,b)$.

Let me leave this preliminary discussion of integrable functions by mentioning that if you want to push the assertion(s) of the fundamental theorem of calculus to apply to more general functions, i.e., in more general situations, then these are the kinds of collections you need to consider.

1.4 The Mean Value Theorem and Rolle's Theorem

These are usually stated for functions $u \in C^0[a,b] \cap \text{Diff}(a,b)$ which, as usual, means continuous on the (finite) closed interval $[a,b]$ and differentiable on the open subinterval (a,b). This explains the (weaker) hypothesis on the function f in Theorem 4.

Theorem 5 *(the mean value theorem) If $u \in C^0[a,b] \cap \text{Diff}(a,b)$, then there exists some $\xi \in (a,b)$ with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 6 *(Rolle's theorem) If $u \in C^0[a,b] \cap \text{Diff}(a,b)$ with*

$$f(b) = f(a),$$

then there exists some $\xi \in (a,b)$ with

$$f'(\xi) = 0.$$

Exercise 23 *Prove Rolle's theorem, and then use Rolle's theorem to prove the mean value theorem. Hints: Show that if a differentiable function has a maximum value at ξ, then $u'(\xi) = 0$. To get the mean value theorem from Rolle's theorem, consider $u(x) - \ell(x)$ where $\ell(x) = mx + b$ is an appropriate affine function.*

With this notation/discussion/review in hand, let us return to our discussion of FTC equations.

2 Existence and Uniqueness

The simplest FTC equation is $y' = 0$. This equation (and FTC equations in general) should not be expected to have a unique solution.

The solution of $y' = 0$, in this case, is $y(x) = c$ for any constant c. More properly, we should say that the **solution set**

$$\Sigma = \{u \in C^1(a,b) : u' = 0\}$$

is the collection of all constant functions. This solution set Σ is isomorphic as a vector space to the vector space $\mathbb{R}^1$ over the field $\mathbb{R}$. As the previous section may have suggested, the collection of functions $C^1(a,b)$ used here is natural in a certain sense, but is used primarily for convenience and other choices are possible. For example, were we to look for functions in Diff(a, b) satisfying $y' = 0$, then the identification of

$$\Sigma_c = \{u \in \text{Diff}(a, b) : u' = 0\},$$

the classically differentiable solutions, with the constant functions may not be so easy. We could also consider even more general solution sets as follows:

$$\Sigma_R = \left\{u \in \underset{loc}{\text{Riem}}(a, b) : -\int_{(a,b)} u\phi' = 0 \text{ for all } \phi \in C_c^\infty(a, b)\right\}$$

where $C_c^\infty(a, b)$ denotes the collection of functions in $C^\infty(a, b)$ having compact support contained in (a, b), that is, there exists some $\epsilon > 0$ for which $\{x \in (a, b) : \phi(x) \neq 0\} \subset (a+\epsilon, b-\epsilon)$. The functions in Σ_R are called the (locally) Riemann integrable **weak solutions** of $u' = 0$. More generally,

$$\Sigma_W = \left\{u \in L^1_{loc}(a, b) : -\int_{(a,b)} u\phi' = 0 \text{ for all } \phi \in C_c^\infty(a, b)\right\}$$

is the collection of weak solutions of $u' = 0$.

Here is a proof for Σ: If $u \in \Sigma$, then given x_1 and x_2 with $a < x_1 < x_2 < b$, we know $u \in C^0[x_1, x_2]$ by Exercise 7. Thus, if $u(x_2) \neq u(x_1)$, then it follows from the mean value theorem that there exists a point ξ between x_1 and x_2 in (a, b) with $u'(\xi) \neq 0$. This contradicts the fact that $u'(\xi) = 0$ and shows u must be constant. □

The same proof works for Σ_c. It does not work for Σ_R or Σ_W. One major advantage of using the classically differentiable functions in $C^1(a, b)$ is that the collection of derivatives of these functions has a nice characterization:

$$\{u' : u \in C^1(a, b)\} = C^0(a, b).$$

But what is

$$\{u' : u \in \text{Diff}(a, b)\}?$$

I don't know a nice characterization of this vector space. Accordingly, let's proceed with Σ for now.

It will also be noted that $L : C^1(a, b) \to C^0(a, b)$ by $Lu = u'$ is a linear function. Thus, our ODE is a question about the linear map L, namely, given $f \in C^0(a, b)$ what is

$$\Sigma = \{u \in C^1(a, b) : u' = f\}.$$

From this point of view, we can recognize the solution set with $f \equiv 0$ mentioned above as the kernel of L. Moreover, the indefinite integral version of the fundamental theorem of calculus gives us existence of solutions: For any $x_0 \in (a, b)$, the function $F : (a, b) \to \mathbb{R}$ by

$$F(x) = \int_{x_0}^{x} f(t)\, dt \tag{4}$$

satisfies $F \in \Sigma$.

Theorem 7 *(uniqueness of C^1 solutions for the general FTC equation) Let $f \in C^0(a, b)$ be given. Then*

1. *If $y \in \Sigma = \{u \in C^1(a, b) : u' = f\}$, then y differs from F by a constant.*

2. *$\Sigma = \{F + y_h : y_h \in \ker(L)\}$ where F is given in (4).*

3. *$\Sigma = \{y_p + y_h : y_h \in \ker(L)\}$ where y_p is any particular solution of $y' = f$.*

4. *For any $x_0 \in (a, b)$ and any $y_0 \in \mathbb{R}$ the initial value problem (IVP)*

$$\begin{cases} y' = f \\ y(x_0) = y_0 \end{cases}$$

has a unique solution.

Proof:

1. If $y \in \Sigma$, then $y - F \in C^1(a, b)$, and $L(y - F) = Ly - LF = y' - F' = 0$. Therefore, $y - F \in \ker(L)$ which is the set of constant functions.

2. Let $y \in \Sigma$, then setting $y_h = y - F$, we have calculated above that $y_h = y - F \in \ker(L)$. Thus, we simply need to notice that

$$y = F + (y - F) = f + y_h.$$

3. Let $y \in \Sigma$, ten setting $y_h = y - y_p$ where y_p is any particular solution, we can compute $y_h' = y' - y_p' = f - f = 0$. Thus, $y_h \in \ker(L)$ and $y = y_p + (y - y_p) = y + y_h$.

4. First note that $y : (a, b) \to \mathbb{R}$ given by

$$y_1(x) = y_0 + F(x) = y_0 + \int_{x_0}^{x} f(t)\, dt$$

is a solution of the IVP. We know that every solution of $y' = f$ differs from y_1 by a constant c. But if y is any solution of the IVP, then we must have

$$c = y(x_0) - y_1(x_0) = y_0 - y_0 = 0. \qquad \square$$

Exercise 24 *Show the reverse inclusion in parts 2 and 3 of the proof of Theorem 7.*

Some aspects of our discussion of the FTC equation and the proof of Theorem 7 apply in a rather more general context:

Theorem 8 *(linear theory: first observations) If $L : V \to W$ is a linear function the associated to every equation*

$$Lv = w$$

with $w \in W$, there is an **associated homogeneous equation**

$$Lv = \mathbf{0} \in W$$

and a well-defined (nonempty) solution set $\ker(L)$.

Given a particular solution v_p of the equation $Lv = w$, the solution set

$$\Sigma = \{v \in V : Lv = w\}$$

may be written as

$$\Sigma = \{v_p + v_h : v_h \in \ker(L)\}.$$

Exercise 25 *Prove Theorem 8.*

A Summary of our discussion of the FTC equation $y' = f$ and the associated initial value problem may be given as follows: The linear function $L : C^1(a,b) \to C^0(a,b)$ is surjective (onto), and given $x_0 \in (a,b)$ there is a (linear) solution operator

$$\sigma : C^0(a,b) \times \mathbb{R} \to C^1(a,b)$$

given by $\sigma[f, y_0] = y$ where

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\, dt.$$

The function σ is a bijection.

3 Less Convenient Assumptions

Were we to consider other natural frameworks in which to consider the FTC equation $y' = f$ our summary would not work out so nicely. There are two obvious alternatives which are both mentioned briefly above. I will attempt to fill in some of the details of one of those here.

3.1 Classically Differentiable Functions

One alternative would be to consider the classically differentiable functions $\text{Diff}(a, b)$ as a source of solutions for $y' = f$. As mentioned above, the collection of derivatives

$$\{u' : u \in \text{Diff}(a, b)\}$$

of classically differentiable functions is not so easy to understand. We should at least know that $C^1(a, b) \subsetneq \text{Diff}(a, b)$. That is, we would like to see a function which is classically differentiable but does not have a continuous derivative. One such example is $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Exercise 26 *Plot the function f either by hand or using mathmatical software. Show that $f \in \text{Diff}(\mathbb{R})\backslash C^1(\mathbb{R})$.*

It does happen to be known that the derivative of any function in $\text{Diff}(a, b)$ must be continuous at uncountably many points that happen to be dense in the interval (a, b). "Uncountably many" means there are more than you can put in one-to-one correspondence with the integers, or alternatively that they can be put in one-to-one correspondence with all the real numbers. "Dense" means that you can find points of continuity arbitrarily close to every point of (a, b). So this suggests the derivative of a function in $\text{Diff}(a, b)$ has many points of continuity. It should be noted, however, that the set of points of discontinuity can (sometimes) also have these same properties. For an interesting example with "many" points of discontinuity, you can look up Volterra's function.

3.2 Weakly Differentiable Functions

A more satisfactory treatment of FTC equations (and a more general version of the fundamental theorem of calculus) may be obtained using **weak derivatives** and **weak solutions**. The foundation for these collections of functions lies in finding a more general notion of integrability—a more general notion of the integral—than that afforded by the **Reimann integral**. An alternative is the **Lebesgue integral** which, in turn, has a definition resting on the ability to measure the **length** of a wider variety of sets than just intervals. We will not, and hopefully do not need to, go into the details of the **measure theory** of sets and the definition of the Lebesgue integral, though for an introduction you may see my notes on integration. It is worthwhile, however, to know about the possibility of a more general integral and some of the things you can do with such a thing. Thus, let us start with the following simple and somewhat vague foundation:

3.2.1 Measurability and Lebesgue integration

We have mentioned the Riemann integral as a limit of Riemann sums above. The main theorem concerning Rieman integrability was Theorem 3, which is called the **Riemann integrability theorem** and states (roughly) that any **continuous** function has a well-defined Riemann integral. We observed that some discontinuous functions may also have a well-defined Riemann integral and thus introduced the collections $\text{Riem}[a, b]$ and $\text{Riem}(a, b)$ of Riemann integrable functions.

In Exercise 13 the functions u_0 and u_1 are seen to be non-Riemann integrable. One way to think of this is in terms of upper and lower Riemann sums.

Exercise 27 *The* **upper Riemann sum** *is defined just like the Riemann sum*

$$\sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1})$$

except that the values $f(x_j^*)$ at the evaluation points $x_j^* \in [x_{j-1}, x_j]$ are replaced by the **supremum** *of the values of f on the interval $[x_{j-1}, x_j]$. The* **supremum** *of any nonempty collection $A \subset \mathbb{R}$ is the* **smallest extended real valued number** *$M \in (-\infty, \infty]$ with $M \geq a$ for every $a \in A$. Such a number is always well-defined and is denoted by*

$$\sup_{a \in A} a \qquad \text{or} \qquad \sup A = \min\{M \in (-\infty, \infty] : M \geq a \text{ for all } a \in A\}.$$

Thus, the upper Riemann sum is

$$\sum_{j=1}^{n} \sup\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1}).$$

This number always exists as an extended real number in $(-\infty, \infty]$.

(a) *Show my claim about the supremum of any nonempty collection of real numbers being well-defined is true. (Hint: You'll need the greatest lower bound property of the real numbers for this.)*

(b) *Define the* **infemum** *of any set of real numbers in a manner similar to that used for the supremum.*

(c) *Using the infemum, define the lower Riemann sum of a function $f : [a, b] \to \mathbb{R}$.*

(d) *Show that the limits*

$$\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\sup\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}) \qquad \text{and} \qquad \lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\inf\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}$$

always exist and satisfy

$$\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\sup\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}) \le \lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\inf\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}).$$

(e) *Show that a function* $f:[a,b]\to\mathbb{R}$ *is Riemann integrable if and only if*

$$\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\sup\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}) = \lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^{n}\inf\{f(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}).$$

(f) *Find*

$$\sum_{j=1}^{n}\sup\{u_k(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1}) \qquad \text{and} \qquad \sum_{j=1}^{n}\inf\{u_k(x): x_{j-1}\le x\le x_j\}(x_j-x_{j-1})$$

for $k=0$ *and* 1 *where* u_0 *and* u_1 *are the functions defined in Exercise 13.*

...More on this later

4 Other Linear ODE

Given the manner in which we have defined ordinary differential equations, you may be wondering where linear equations like $y''=-y$ with solutions

$$\Sigma=\{a\cos x+b\sin x: a,b\in\mathbb{R}\}$$

fit in. How does this ODE $y''=-y$ specify a single derivative? I will answer that question soon, but for now let's stick with the framework of first order equations and consider the somewhat broader set of ODE of the form

$$Ly = y' + p(x)y = g(x) \tag{5}$$

for the unknown function $y \in C^1(a,b)$ of which the FTC equations considered above are the special case with $p \equiv 0$ and $f = g$. Equations of the form (5) do fit into our framework if we write

$$y' = g(x) - p(x)y = f(x,y) \tag{6}$$

where now the derivative is specified by a function $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ having the special quasi-linear (i.e., affine in y) form $f(x,y) = g(x) - p(x)y$.

If we stick to the elementary framework in which $f \in C^0(\mathbb{R} \times (a,b))$ and $y \in C^1(a,b)$, then the theory of this equation is nice and simple. Perhaps the most important aspect of that theory is the existence and uniqueness theorem:

Theorem 9 *(linear existence and uniqueness—first order) If $p, g \in C^0(a,b)$, $x_0 \in (a,b)$, and $y_0 \in \mathbb{R}$, then the IVP*

$$\begin{cases} Ly = g \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y \in C^1(a,b)$.

The result follows from an easy application of linear theory along with application of the fundamental theorem of calculus, or equivalently the theory of FTC equations above:

The **associated homogeneous equation** is

$$Ly = y' + py = 0. \tag{7}$$

This equation is easy to solve thanks to the existence of a **positive integrating factor**

$$\mu(x) = e^{\int_{x_1}^{x} p(t)\,dt}.$$

The integrating factor is, naturally, only determined up to a positive multiple depending on the initial point of integration which may be chosen for convenience. For this reason, the expression for μ is often written as

$$\mu(x) = e^{\int^{x} p(t)\,dt}$$

with no lower limit of integration. In any case, the key fact is that $\mu \in C^1(a,b)$ with $\mu'(x) = p(x)\mu(x)$. This follows from the fundamental theorem of calculus and the chain rule. In particular, (7) can be written as

$$(\mu y)' = 0$$

which is the homogeneous FTC equation. Thus, the kernel of L when $Ly = y' + py$ is given by

$$\ker(L) = \left\{ \frac{c}{\mu} : c \in \mathbb{R} \right\} = \left\{ c e^{-\int_{x_1}^{x} p(t)\,dt} : c \in \mathbb{R} \right\}.$$

Exercise 28 *Note that given any positive number m and any* $x_1 \in (a, b)$*, the function*

$$\mu(x) = me^{\int_{x_1}^x p(t)\,dt}$$

works as an integrating factor to solve (7). Is it always possible to choose an initial point $t_1 \in (a, b)$ *such that*

$$me^{\int_{x_1}^x p(t)\,dt} = e^{\int_{t_1}^x p(t)\,dt},$$

that is given $x_1 \in (a, b)$ *fixed, are the two sets*

$$\left\{\mu \in C^1(a, b) : \mu(x) = me^{\int_{x_1}^x p(t)\,dt} \text{ for some } m > 0\right\}$$

and

$$\left\{\mu \in C^1(a, b) : \mu(x) = e^{\int_{t_1}^x p(t)\,dt} \text{ for some } t_1 \in (a, b)\right\}$$

equal?

Exercise 29 *Recall the discussion of (6) in which we wrote* $f(x, y) = g(x) - p(x)y$*. Show the regularity hypothesis* $f \in C^0((a, b) \times \mathbb{R}$ *is equivalent to the regularity hypothesis* $p, g \in C^1(a, b)$ *from Theorem 9.*

General linear theory now tells us the solution set for $Ly = g$ is given by

$$\Sigma = \{y \in C^1(a, b) : y' + py = g\} = \{y_p + y_h : y_h \in \ker(L)\}$$

where y_p is any one particular solution of $y' + py = g$.

A particular solution is obtained from the familiar modification of our approach to the homogeneous equation:

$$(\mu y_p)' = \mu g$$

is an FTC equation. Integrating both sides from x_0 (fixed in (a, b)) to $x \in (a, b)$ and assuming $y_p(x_0) = 0$ (since we are seeking any one particular solution), we get

$$\mu y_p = \int_{x_0}^x \mu(\xi)g(\xi)\,d\xi = \int_{x_0}^x e^{\int_{x_1}^\xi p(t)\,dt} g(\xi)\,d\xi.$$

That is,

$$\begin{aligned} y(x) &= e^{-\int_{x_1}^x p(t)\,dt}\left(y_0 + \int_{x_0}^x e^{\int_{x_1}^\xi p(t)\,dt} g(\xi)\,d\xi\right) \\ &= y_0 e^{-\int_{x_1}^x p(t)\,dt} + \int_{x_0}^x e^{\int_x^\xi p(t)\,dt} g(\xi)\,d\xi. \end{aligned}$$

This should be a formula with which you are familiar, and you should be able to use it.

As in the case of the FTC equation, $\sigma : \mathbb{R} \times C^0(a,b) \to C^1(a,b)$ by

$$(y_0, g) \mapsto y$$

is a linear isomorphism of vector spaces.

Exercise 30 *Note that the linear isomorphism $\sigma = \sigma_p$ depends (primarily) on one coefficient function $p \in C^0(a,b)$ with σ_0 giving the solution operator for the FTC equation.*

(a) *On what else does σ_p depend?*

(b) *If $x_0, x_1 \in (a,b)$ are fixed, show σ_p and $\sigma_{\tilde{p}}$ are different isomorphisms if $p \neq \tilde{p}$. Thus, $p \mapsto \sigma_p$ gives an injection*

$$\Psi : C^0(a,b) \to \mathcal{L}(\mathbb{R} \times C^0(a,b) \to C^1(a,b))$$

where $\mathcal{L}(\mathbb{R} \times C^0(a,b) \to C^1(a,b))$ is the vector space of all linear functions from $\mathbb{R} \times C^0(a,b)$ to $C^1(a,b)$.

(c) *Is Ψ onto?*

(d) *Is Ψ linear?*

(e) *Let $p_1 : (a,b) \to \mathbb{R}$ be the constant function with $p_1(x) \equiv 1$. Write down the expressions for $\sigma_0(y_0, g)$, $\Psi(p_1)(y_0, g)$, and $\Psi(cp_1)(y_0, g)$.*

Exercise 31 *Consider the singular linear ordinary differential operator $L : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ given by*

$$Ly = a(x)y' + b(x)y$$

for $a, b \in C^0(\mathbb{R})$ fixed and satisfying $\{x \in \mathbb{R} : a(x) = 0\} = \{0\}$ and $0 \notin \{x \in \mathbb{R} : b(x) = 0\}$. Show that given any $g \in C^0(\mathbb{R})$ and any $\epsilon > 0$, there does **not exist** *a solution $y \in C^1(-\epsilon, \epsilon)$ of*

$$Ly = g.$$

Exercise 32 *State and prove an existence and uniqueness theorem for the singular linear first order ODE $tx' = 0$ for $x = x(t)$.*

5 The Spaces $C^0(\mathbb{R}^n \to \mathbb{R}^n)$ and $C^1((a,b) \to \mathbb{R}^n)$.

$C^0(\mathbb{R}^n \to \mathbb{R}^n)$ is the space of **continuous vector fields**. A function $\mathbf{F} \in C^0(\mathbb{R}^n \to \mathbb{R}^n)$ has the form

$$\mathbf{F} = (f_1, f_2, \dots, f_n) \qquad \text{with} \qquad f_j \in c^0(\mathbb{R}^n) \quad \text{for} \quad j = 1, 2, \dots, n.$$

Remember what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be in $C^0(\mathbb{R}^n)$:

Given any $\mathbf{x}_0 \in \mathbb{R}^n$ and any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon \qquad \text{whenever } |\mathbf{x} - \mathbf{x}_0| < \delta.$$

Thus, we are requiring this condition to hold for each component function $f = f_j$ ($j = 1, 2, \dots, n$).

Exercise 33 *Show that the continuity condition above holding for $f = f_j$ ($j = 1, 2, \dots, n$) is equivalent to the condition*

Given any $\mathbf{x}_0 \in \mathbb{R}^n$ and any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)| < \epsilon \qquad \text{whenever} \qquad |\mathbf{x} - \mathbf{x}_0| < \delta.$$

These vector valued functions of a vector variable might also look like

$$\mathbf{v} = (v_1, v_2, \dots, v_n) \qquad \text{or} \qquad \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

$C^1((a,b) \to \mathbb{R}^n)$ is the collection of **continuously differentiable parametric curves** with image in $\mathbb{R}^n$. These have the form

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \qquad \text{or} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $x_j \in C^1(a,b)$ for each $j = 1, 2, \dots, n$. In particular,

$$\mathbf{x}' = (x_1', x_2', \dots, x_n') \qquad \text{or} \qquad \mathbf{x} = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$$

Exercise 34 *If* $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$*, then for each* $t \in (a,b)$

$$\mathbf{x}' = \lim_{h\to 0} \frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h}.$$

Here is a short list of ODEs:

$$\mathbf{x}' = \mathbf{F}(t). \tag{8}$$

The ODE (8) is an **FTC system**. Simple assumptions are $\mathbf{F} \in C^0((a,b) \to \mathbb{R}^n)$ and $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$.

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}). \tag{9}$$

The ODE (9) is an **autonomous system**. Simple assumptions are $\mathbf{F} \in C^0(\mathbb{R}^n \to \mathbb{R}^n)$ and $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$.

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t). \tag{10}$$

The ODE (10) is an **general system of ODEs**. Simple assumptions are $\mathbf{F} \in C^0(\mathbb{R}^n \times (a,b) \to \mathbb{R}^n)$ and $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$.

Alternative assumptions for each of the ODEs above are obtained by taking the codomain of $\mathbf{x}$ to be the complex numbers.

Exercise 35 *If* $\mathbf{x}$ *is assumed to be in* $C^1((a,b) \to \mathbb{C}^n)$ *in each of the ODEs above, what kind of function is* $\mathbf{F}$ *(by implication)?*

We conclude this section by writing out the general autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, so you can see that it looks (roughly) like a system of single autonomous equations of the form $x' = f(x)$:

$$\begin{cases} x_1' &= f_1(x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, x_2, \ldots, x_n), \end{cases}$$

Exercise 36 *Write out the general system of ODEs* $\mathbf{x}' = \mathbf{F}(\mathbf{x}, t)$*.*

6 Autonomous Single Equations

Let's consider a single ODE of the form

$$x' = f(x).$$

We know that if $f(x)$ has the form $f(x) = -ax + b$ for some constants a and b, then we have a linear equation, and we can determine everything about it using linear theory.

Exercise 37 *Assuming* $x' = -ax + b$ *for constants a and b, identify the appropriate linear operator L for the ODE and find the kernel of L. Find a particular solution and solve the ODE in general.*

Generally, f will not be an affine function. There is still some general theory associated with autonomous equations $x' = f(x)$ and much of it applies also to autonomous systems $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

The first thing to consider with an autonomous system is the possibility of **equilibrium points**. Equilibrium points are constant solutions of the ODE. There is one associated with every value x_* of the equation

$$f(x_*) = 0.$$

That is, the equilibrium points are **zeros** of the specification[2] function f.

Exercise 38 *If* $f(x_*) = 0$*, then* $x : \mathbb{R} \to \mathbb{R}$ *by* $x(t) \equiv x_*$ *is a solution of* $x' = f(x)$*. (Verify that this is the case.) When you understand this assertion, formulate what it means to have an equilibrium solution for a system* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$*, and formulate an analogous assertion concerning constant solutions for systems.*

While equilibrium solutions are defined for all "time," it is not true that all solutions of autonomous systems are defined for all time, even if the specification function is smooth. Here is a simple version of the main theorem on existence for nonlinear autonomous ODE:

Theorem 10 *(nonlinear existence and uniqueness theorem—autonomous ODE) If* $f \in C^1(\mathbb{R})$*, then given any* $t_0 \in (a, b)$ *and any* $x_0 \in \mathbb{R}$*, there exists some* $\delta > 0$ *such that the IVP*

$$\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution $x \in C^1(t_0 - \delta, t_0 + \delta)$*.*

Essentially the same result holds for systems. The statement is as follows:

If $\mathbf{F} \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$*, then given any* $t_0 \in (a, b)$ *and any* $\mathbf{x}_0 \in \mathbb{R}^n$*, there exists some* $\delta > 0$ *such that the IVP*

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution $\mathbf{x} \in C^1((t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n)$*.*

[2]Note carefully that f is not an "inhomogeneity" here. The terms "homogeneous" and "inhomogeneous" only apply when you have linear equations, and autonomous equations are ususally not linear.

The following example is fundamental in illustrating the nonlinear existence and uniqueness theorem:

$$\begin{cases} y' = y^2 \\ y(t_0) = y_0 \end{cases}$$

Notice there is no singularity in the equation—there is no reason to believe, aside from the nonlinearity that there should be any problem with the regularity and long time existence of solutions.

The equilibrium solution is $y_* = 0$. If $y_0 \neq 0$, then we may locally divide by $y = y(t)$ (if there is a solution). Then we can integrate both sides from t_0 to t to obtain

$$\int_{t_0}^{t} \frac{y'(\tau)}{[y(\tau)]^2}\, d\tau = t - t_0.$$

Changing variables in the integral on the left with $\xi = y(\tau)$ so that $d\xi = y'(\tau)\, d\tau$, we have

$$\int_{y_0}^{y} \frac{1}{\xi^2}\, d\xi = t - t_0.$$

That is,

$$-\frac{1}{y} + \frac{1}{y_0} = t - t_0 \qquad \text{or} \qquad y(t) = \frac{1}{\frac{1}{y_0} - (t - t_0)} = \frac{y_0}{1 - y_0(t - t_0)}.$$

Evidently, this solution has a singularity—finite time blow-up—at time

$$t_1 = \frac{1}{y_0} + t_0.$$

If y_0 is positive, then the finite blow-up time, is future to the initial time $t = t_0$. In particular, we can only say

$$y \in C^1\left(-\infty, t_0 + \frac{1}{y_0}\right),$$

and there is no obvious way to guess this without solving the ODE. If y_0 is negative, then there is a negative time blow-up. Still, even though $f \in C^\infty(\mathbb{R})$, we don't get $y \in C^{(}\mathbb{R})$.

We should return to this example after we discuss **phase diagrams** (below). Before we take up that discussion, I want to make a simple observation concerning the time dependence/relation for autonomous ODEs and give some definitions.

Obviously, the existence and uniqueness theorem is somewhat disappointing in regard to existence. Nevertheless, if we have existence, the theorem gives a solid conclusion with regard to uniqueness:

Theorem 11 *(global uniqueness of solutions for nonlinear ODE) If* $\mathbf{F} \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ *and* $\mathbf{y} \in C^1(a, b)$ *is a solution of the IVP*

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

for some $t_0 \in (a, b)$ *and* $\mathbf{x}_0 \in \mathbb{R}^n$*, then* $\mathbf{y}$ *is the unique solution of this problem in* $C^1(a, b)$*.*

Exercise 39 *Use the local existence and uniqueness theorem to prove the global uniqueness theorem.*

Furthermore, an interesting "invariance in starting time" applies to autonomous ODEs. If $\mathbf{y}_0 \in C^1(\mathbb{R} \to \mathbb{R}^n)$ is a solution of the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases}$$

then $\mathbf{y}(t) = \mathbf{y}_0(t + t_1)$ is a solution of

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(t_0 - t_1) = \mathbf{x}_0, \end{cases} .$$

This doesn't mean all solutions of the ODE are obtained by shifting in time, but it does mean that any two solutions taking on the same value at any time are essentially the same solution—they just differ by a shift in time.

Exercise 40 *Consider the autonomous ODE* $x' = x$*. Draw all solutions in* **solution space***. (Solution space, in this case, is the* t, x*-plane where the graphs of solutions* $x = x(t)$ *are plotted.) Notice that any pair of solutions* x_1 *and* x_2 *with any common value* $x_1(t_1) = x_2(t_2) = x_0$ *are shifts in time of one another. Notice also that, up to a shift in time, the behavior of every solution satisfying* $x(t_1) = x_0$ *for some time t may be seen in the solution satisfying* $x(0) = x_0$*.*

The ODE in Exercise 40 is linear and has solutions existing for all time. Generally, we must make that an assumption, but it does happen sometimes.

Definition 2 *An equilibrium point* $\mathbf{x}_*$ *for the autonomous ODE* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *is* **stable** *if for any* $r > 0$*, there is some* $\delta > 0$ *for which the following holds:*

If $\mathbf{x}_0 \in B_\delta(\mathbf{x}_0)$*, then*

1. *The IVP*

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

has a (unique) solution $\mathbf{y}$ *defined for all* $t \in \mathbb{R}$, *and*

2. $\mathbf{y}(t) \in B_r(\mathbf{x}_*)$ *for all* $t \geq 0$.

Definition 3 *An equilibrium point* $\mathbf{x}_*$ *for the autonomous ODE* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *is* **asymptotically stable** *if there is some* $\delta > 0$ *for which the following holds:*

If $\mathbf{x}_0 \in B_\delta(\mathbf{x}_0)$, *then*

1. *The IVP*

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

has a (unique) solution $\mathbf{y}$ *defined for all* $t \in \mathbb{R}$, *and*

2.

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{x}_*.$$

Exercise 41 *Show that every asymptotically stable equilibrium point is stable.*

Definition 4 **Phase space** *is the term used to refer to the codomain of solutions in an autonomous ODE* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. *Thus, if the equation is for* $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$, *then the phase space for the system is* $\mathbb{R}^n$. *A phase space diagram is an explantion of the orbit structure of solutions in phase space. An* **orbit** *is a set containing all images of a solution:*

$$\mathcal{O} = \{\mathbf{x}(t) : t \in \mathbb{R}\}$$

where $\mathbf{x} \in C^1(\mathbb{R} \to \mathbb{R}^n)$ *and* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. *When we consider phase space, we always require* $\mathbf{F}$ *to have adequate regularity to ensure the uniqueness of solutions. Thus, we assume solutions are determined by an initial value* $\mathbf{x}(0) = \mathbf{x}_0$. *Accordingly, orbits are determined by the initial value as well, and If* $\mathbf{x}(0) = \mathbf{x}_0$, *we write* $\mathcal{O} = \mathcal{O}(\mathbf{x}_0)$ *for the orbit passing through* $\mathbf{x}_0$.

Technically, we sometimes consider phase space and orbits for solutions whose domain is some open interval (a, b) rather than the entire real line $\mathbb{R}$. The definition above requires some minor modifications in this case.

Exercise 42 *Show that if* $\mathbf{x}(t_0) = \mathbf{x}_0$, *then*

$$\{\mathbf{x}(t) : t \in \mathbb{R}\} = \mathcal{O}(\mathbf{x}_0).$$

Exercise 43 *Show, more generally, that if two orbits have a point in common or "cross" then they are the same orbit.*

7 Autonomous Systems

The study of equilibrium points and stability becomes rather more interesting when the dimension of a system is greater than one. In particular, the case of two-dimensional autonomous systems

$$\mathbf{x}' = \mathbf{F}(\mathbf{x})$$

for $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ (or alternatively $\mathbf{x} : (a, b) \to \mathbb{R}^2$, though the identification of the domain interval (a, b) is often ambiguous in this context) offers some reasonable idea of what may be involved in general. This situation is rather analogous to the study of linear functions $L : \mathbb{R}^2 \to \mathbb{R}^2$ in linear algebra and, in fact, we will have opportunity to revisit our consideration of this topic.

Here is an example of a (marginally interesting) autonomous system for two imagined populations:

$$\begin{cases} r' = 0.5r(1 - 0.2r - f) \\ f' = 0.1f(-1 + 2r - 0.3f). \end{cases} \tag{11}$$

When rabbits, whose population is represented here (perhaps measured in the thousands) by the quantity r, are left to themselves, they tend to increase in numbers. In this model, that growth in population is exponential according to $r' = 0.5r$. The presence of predator foxes, whose population is represented in this model by $f : \mathbb{R} \to \mathbb{R}$, leads to a decrease in the rabbit population. This is modeled by the fact that for $f > 1 - 0.2r$ the quantity $r' = r(1 - 0.2r - f) < 0$. Foxes, according to this model, are in the somewhat precarious position of experiencing inevitible extinction in the absence of rabbits. The presence of many tasty rabbits, however, leads also to an increasing fox population.

Exercise 44 *Consider the single autonomous ODE for rabbits in the absence of foxes according to (11). Plot and interpret the phase line diagram for the rabbit population in this case. Note that this is a logistic equation.*

Exercise 45 *Consider the single autonomous ODE for foxes in the absence of rabbits according to (11). Plot and interpret the phase line diagram for the rabbit population in this case. Is this a logistic equation?*

Notice that the phase line diagrams from Exercises 44 and 45 may be transferred directly to the appropriate axes of the phase plane diagram for the system (11).

Exercise 46 *Plot the orbits in the r, f-phase plane corresponding to solutions satisfying the initial condition $f_0 = 0$. The existence and uniqueness theorem for systems tells us that no other orbit can cross these orbits.*

The actual coefficients I have chosen are presumably not very realistic, but our considerations are only for illustrative purposes; no rabbits nor foxes have been directly harmed in the making of these notes. (I will leave that activity to my students.) Also, the model is presumably not a very realistic one, but the study of population dynamics (or more broadly mathematical biology) is a rather crude field in which models like this one are quite common.[3] Whether the crudeness of the study of population dynamics leads to greater (actual) damage of wildlife populations, or lesser damage is not clear. What does seem clear is that we have the dubious privilege of seeing the simultaneous increase of technology (or more properly technocracy), sophistication of mathematical and scientific models, absurd hubris in a certain ever growing number of individual humans, and the unprecedented destruction of the environment and many of the biological inhabitants of the earth (including humans).

A humorous (though still quite sad) illustration of this complicated phenomenon was presented recently by the Atlanta Journal Constitution. An article dated March 31, 2020 presented the model findings of Emory "infectious disease expert" Carlos del Rio. Dr. del Rio was using "the best models we have" and predicted, among other things, that "there should be no more deaths due to the carona virus after June 9." This was one of many predictions which were all equally absurd and meaningless.

It is not unreasonable to expect that humans using models **know** the limitations of the modeling—and in particular if the model predictions are meaningless—**before** predictions are used as the basis of destructive action. Apparently, however, this is almost universally not the case, and it is an extremely interesting question as to why this inexplicably destructive behavior persists among humans. Given that technology (or perhaps more properly technocracy) and advances in scientific understanding are so strongly linked to the destruction of the environment, living organisms, and even other humans, why does it seem nearly universally assumed that the same behavior is a desirable "good" leading to essentially the precise opposite of the pretended outcomes of "saving lives," "saving the environment," "solving problems," and generally improving the situation?

In any case, let's see what we can say about the future of our model rabbits and foxes. The function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ in this case determines a vector field on the plane. One can plot this vector field either by hand or using mathematical software. Let's see what we can tell directly. We know the vanishing of the vector field corresponds to the presence of constant solutions, or equilibria in the phase plane. These occur at $(r_*, f_*) = (0, 0)$ when both populations start (and remain) zero, or when $f \equiv f_* = 0$ and we obtain from the first equation in (11) $r_* = 5$. There is also a non-meaningful equilibrium at $r_* = 0$, $f_* = -10/3$. Finally, if we seek an equilibrium in which both populations are modeled with a positive

[3] Look up "predator-prey" or 'Lotke-Volterra" equations. You will see these are "frequently used" and are even simpler than the model I have used here.

value, we find $(r_*, f_*) = (65, 90)/103$.

Nullclines

The curves (which in this instance are straight lines) where one of the components of the specification field vanish have special significance. Along the r-axis, for example, we have $f \equiv 0$ and, consequently $f' \equiv 0$. As mentioned above, existence and uniqueness tells us that no orbit can cross the axes. The line $-1 + 2r - 0.3f = 0$, or $20r - 3f = 10$, in phase space is also a nullcline but is not along an orbit. Every orbit, however, that crosses the line $20r - 3f = 10$ must do so having a horizontal tangent at the point of crossing. Furthermore, the two lines $f = 0$ and $20r - 3f = 10$ comprise the only set of points where an orbit can have a horizontal tangent. Thus, all other points on all other orbits are either "moving" up or down—and can not change direction in this sense, except when crossing these two lines.

The **vertical nullclines** given by the f-axis $r = 0$ and the line $r + 5f = 5$ along with the **horizontal nullclines** $f = 0$ and $20r - 3f = 10$ discussed above partition the phase plane into eleven regions:

$$
\begin{aligned}
\text{I}: &\quad \{(r, f) : r > 0 \text{ and } f < \min\{0, 10(2r-1)/3, 1 - r/5\}\}\\
\text{II}: &\quad \{(r, f) : r > 5 \text{ and } 1 - r/5 < f < 0\}\\
\text{III}: &\quad \{(r, f) : r > 65/103 \text{ and } \max\{1 - r/5, 0\} < f < 10(2r-1)/3\}\\
\text{IV}: &\quad \{(r, f) : 1/2 < r < 5 \text{ and } 0 < f < \min\{10(2r-1)/3, 1 - r/5\}\}\\
\text{V}: &\quad \{(r, f) : r > 0 \text{ and } f > \max\{10(2r-1)/3, 1 - r/5\}\}\\
\text{VI}: &\quad \{(r, f) : 0 < f < 1 \text{ and } 0 < r < \min\{f(1-f), (3f+10)/20\}\}\\
\text{VII}: &\quad \{(r, f) : 0 < r < 1/2 \text{ and } 10(2r-1)/3 < f < 0\}\\
\text{VIII}: &\quad \{(r, f) : r < 0 \text{ and } f > 1 - r/5\}\\
\text{IX}: &\quad \{(r, f) : r < 0 \text{ and } 0 < f < 1 - r/5\}\\
\text{X}: &\quad \{(r, f) : r < 0 \text{ and } 10(2r-1)/3 < f < 0\}\\
\text{XI}: &\quad \{(r, f) : r < 0 \text{ and } f < 10(2r-1)/3\}.
\end{aligned}
$$

Exercise 47 *Draw these eleven regions in the r, f-phase plane and indicate the "compass direction" associated with each region. For example, in region I the vector field at every point is in a direction "up and to the right." This may be seen in different ways. For example, because a portion of the horizontal nullcline $f = 0$ (with value $r < 5$ where the logistic form of the corresponding ODE for rabbits in the absence of foxes dictates an increasing rabbit population) lies along the boundary of region I, we know the vector field* *in region I must point to the right. Similarly, the vector field along the f-axis for very negative f points "up," so the same must be true in region I.*

Alternatively, $\mathbf{F}(1,-1)^T = (90,27)^T/100$ *is pointing up and to the right and* $(1,-1)^T$ *is in region I, so all other points in region must share the same direction.*

The regions of Exercise 47 give some general information about the future values of r and f for given starting populations r_0 and f_0. For example, if (r_0, f_0) is in one of the regions III, V, VI, or IV, then the solution must generate an orbit which moves the population point $(r, f) = (r(t), f(t))$ across the nullclines from one of these regions to the next in the listed order (and then from region IV to region III repeating the pattern). Thus, the orbits of interest "cycle" around the equilibrium point $(90, 27)/100$, and the local behavior of orbits near this equilibrium point is of particular interest.

Again, one may attempt to use mathematical software to determine the behavior in question. We will take a different approach.

7.1 Linear Constant Coefficent Systems

Let us briefly consider systems of the form $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ where A is a constant coefficient matrix. The linear existence and uniqueness theorem gives us that the domain of all solutions is all of $\mathbb{R}$, so we have here a dynamical system.

You may recall that we mentioned briefly in our study of linear functions $L : \mathbb{R}^2 \to \mathbb{R}^2$ certain **canonical forms**. These were given by multiplication $L\mathbf{x} = A\mathbf{x}$ by matrices A of the forms

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} = \sqrt{\lambda^2+\mu^2}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

These are called the real canonical forms. They are **diagonal**, **Jordan form**, and **rotational form**. Taken over the complex numbers, there are only two canonical forms: Diagonal and Jordan form.

Hopefully, you picked up on (or were convinced) of the following

1. A linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication $L\mathbf{x} = A\mathbf{x}$ where A is one of the canonical forms is relatively easy to understand.

2. Not every linear function is as simple as one with canonical form, and the complication has (roughly speaking) to do with eigenvectors which are not orthogonal.

In short, not every linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ is equivalent to one with canonical form. Some, however, are. In particular, Boas emphasizes linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\mathbf{x} = A\mathbf{x}$ where A is a **real symmetric matrix**, i.e., $A^T = A$. These transformations are equivalent, up to a rotation, to a diagonal matrix.

There is another relation between matrices called **similarity**. We also mentioned this briefly, though we may not have used this name. Here is the main result:

Theorem 12 *Every linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\mathbf{x} = A\mathbf{x}$ with respect to the standard matrix may be expressed as*

$$L\mathbf{x} = Q^{-1}\Lambda Q\mathbf{x}$$

where Q is an invertible 2×2 matrix and Λ is a matrix with one of the three real canonical forms.

The matrices A and $\Lambda = QAQ^{-1}$ are said to be **similar**. *Moreover, the form of the matrix Λ is uniquely determined by A, though the matrix Q is not uniquely determined. In particular, A cannot be similar to two different canonical forms.*

The conjugacy relations in Theorem 12 are illustrated in the following matrix/mapping diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \downarrow Q & & \downarrow Q \\ \mathbb{R}^2 & \xrightarrow{\Lambda} & \mathbb{R}^2 \end{array}$$

Note that Theorem 12 is not (quite) saying the matrix Λ is unique. Some choices with regard to the matrix Q are possible and, for example, every matrix A which is similar to the diagonal matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

is also similar to the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Such a matrix A, however, cannot be similar to a Jordan form matrix or a rotational form matrix.

As we should have learned, similarity does not capture the geometric mapping properties of a linear function L completely. It does capture some properties of L, however, and most importantly similarity is adequate (and very useful) for certain **computations**. In particular, computing a matrix exponentiation is realatively easy using similarity to a canonical form. This brings us to the connection wtih ODEs.

Say you want to solve the constant coefficient linear system

$$\mathbf{x}' = A\mathbf{x}$$

for $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$.

Exercise 48 *Say $A = Q^{-1}\Lambda Q$ where Λ is a matrix with one of the canonical forms. Show that given a solution $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ of $\mathbf{x}' = A\mathbf{x}$, the function $Q\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ is a solution of*

$$\mathbf{y}' = \Lambda\mathbf{y}.$$

It follows from this exercise that if you can solve $\mathbf{y}' = \Lambda y$ where Λ is any matrix with one of the canonical forms (and this is relatively easy to do), then you can solve the similar system $\mathbf{x}' = A\mathbf{x}$.

Exercise 49 *Solve $\mathbf{x}' = \Lambda\mathbf{x}$ if Λ has one of the canonical forms.*

The only "loose end" still to be addressed is how to find the change of basis matrix Q. Let us deal with this question in the two "easy" cases now. We can tackle the remaining case in the next section.

Here are the crucial observations:

1. The matrix A is **diagonalizable**, i.e., similar to a diagonal matrix, if and only if, there exists a **basis for** $\mathbb{R}^2$ **consisting of eigenvectors** of $L\mathbf{x} = A\mathbf{x}$.

2. The matrix A is similar to a **Jordan form** matrix if and only if there exists a single eigenvalue **and** the associated eigenspace is one dimensional.

3. The real matrix A is similar to a rotational form matrix if and only if the eigenvalues of $L\mathbf{x} = A\mathbf{x}$ are nontrivially complex, i.e., if there does not exist any real eigenvalue.

Exercise 50 *Give an example of a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ with only one real eigenvalue which is* **not** *similar to a Jordan shear transformation.*

If one has a basis $\{\mathbf{v}, \mathbf{w}\}$ of eigenvectors for $L\mathbf{x} = A\mathbf{x}$, and the eigenvalue-eigenvector pairs are written as $(\lambda_1, \mathbf{v})$ and $(\lambda_2, \mathbf{w})$, then Q is the matrix whose **inverse** has columns $\mathbf{v}$ and $\mathbf{w}$:

$$Q^{-1} = \begin{pmatrix} | & | \\ \mathbf{v} & \mathbf{w} \\ | & | \end{pmatrix}. \tag{12}$$

Note that Q is determined by the mapping conditions $\mathbf{v} \mapsto \mathbf{e}_1$ and $\mathbf{w} \mapsto \mathbf{e}_2$.

Exercise 51 *Based simply on the mapping conditions* $\mathbf{v} \mapsto \mathbf{e}_1$ *and* $\mathbf{w} \mapsto \mathbf{e}_2$ *for* Q *and the mapping conditions* $\mathbf{e}_1 \mapsto \mathbf{v}$ *and* $\mathbf{e}_2 \mapsto \mathbf{w}$ *for* Q^{-1}*, compute*

$$QAQ^{-1}\mathbf{e}_1 \qquad \text{and} \qquad QAQ^{-1}\mathbf{e}_2$$

(where $(\lambda_1, \mathbf{v})$ *and* $(\lambda_2, \mathbf{w})$ *are eigenvalue-eigenvector pairs for* $L\mathbf{x} = A\mathbf{x}$*). These vectors you have computed are the columns of the diagonal form* Λ*.*

The other easy case is when Λ has the Jordan form. We may assume one eigenvector $\mathbf{v}$ generating the one dimensional eigenspace

$$\{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \lambda\mathbf{x}\}.$$

In this case, it is always possible to find a solution of

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

The vector $\mathbf{w}$ is called a generator of the **cyclic subspace** associated with A. Note that $(A - \lambda I)^2\mathbf{x} \equiv \mathbf{0}$. In this case, we take the basis $\{\mathbf{v}, \mathbf{w}\}$ to construct the change of basis matrix Q according to the same formula given in (12).

Exercise 52 *Prove* $(L - \lambda I)^2 \equiv 0$ *in the Jordan form case with eigenvalue* λ*. Compute the columns* $QAQ^{-1}\mathbf{e}_1$ *and* $QAQ^{-1}\mathbf{e}_2$ *of the Jordan canonical form matrix* Λ*.*

7.2 Linearization

Given an equilibrium point $\mathbf{x}_*$ of a nonlinear autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is, to zero order near $\mathbf{x} = \mathbf{x}_*$ approximated by

$$\mathbf{F}(\mathbf{x}) \equiv \mathbf{0}.$$

This, of course, does not tell you much about the behavior of solutions (and orbits) hear the equilibrium point $\mathbf{x}_*$. In order to get some useful information about the local behavior of solutions near an equilibrium point $\mathbf{x} = \mathbf{x}_*$, we can try a **first order approximation** of the vector field $\mathbf{F}$. Sometimes this approach works.

In order to start we need a version of Taylor's first order approximation formula for vector valued functions of a vector variable. To make a long story short, this formula is

$$\mathbf{F}(\mathbf{x}) \sim \mathbf{F}(\mathbf{x}_0) + D\mathbf{F}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

In the case of an equilibrium point $\mathbf{x}_0 = \mathbf{x}_*$ this becomes

$$\mathbf{F}(\mathbf{x}) \sim D\mathbf{F}(\mathbf{x}_*)(\mathbf{x} - \mathbf{x}_*). \tag{13}$$

I still haven't explained the derivative $D\mathbf{F}(\mathbf{x}_0)$, or more immediately of interest $D\mathbf{F}(\mathbf{x}_*)$, so let me try to do that.

7.3 The total derivative of a function of several variables

The derivative, sometimes called the **total derivative** of a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix of partial derivatives

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$$

where $\mathbf{F} = (f_1, f_2, \ldots, f_m)$ has m component functions each of which is a function of n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In the case $m = 1$ of a real valued function of several variables, this derivative is sometimes called the gradient. In the case of one independent variable, the case of ODEs, this derivative is called the velocity vector.

Associated with this matrix is a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$L(\mathbf{v}) = D\mathbf{F}(\mathbf{x}_0)\mathbf{v}.$$

This linear function is called the **differential** and is often denoted by $L = d\mathbf{F}$ or $d\mathbf{F}_{\mathbf{x}_0}$. Thus, Taylor's first order approximation formula may be written as

$$\mathbf{F}(\mathbf{x}) \sim \mathbf{F}(\mathbf{x}_0) + d\mathbf{F}_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) \qquad \text{for } \mathbf{x} \text{ close to } \mathbf{x}_0.$$

7.4 The associated linear ODE

Returning to our consideration of the ODE $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ near an equilibrium point $\mathbf{x} = \mathbf{x}_0$, the approximation (13) suggests consideration of the ODE

$$\mathbf{y}' = D\mathbf{F}(\mathbf{x}_*)(\mathbf{y} - \mathbf{x}_*)$$

or equivalently,

$$\mathbf{y}' = D\mathbf{F}(\mathbf{x}_*)\mathbf{y}.$$

This latter constant coefficient linear system of ODEs is usually referred to as the **linearized system**.

Exercise 53 *Why are the linear constant coefficient systems* $\mathbf{y}' = D\mathbf{F}(\mathbf{x}_*)(\mathbf{y} - \mathbf{x}_*)$ *and* $\mathbf{y}' = D\mathbf{F}(\mathbf{x}_*)\mathbf{y}$ *equivalent?*

Let's take a look at how this can be used near the population equilibrium point $(r_*, f_*) = (65, 90)/103$ for modeling rabbits and foxes. In that case we had

$$\mathbf{F}\begin{pmatrix} r \\ f \end{pmatrix} = \begin{pmatrix} 0.5r(1 - 0.2r - f) \\ 0.1f(-1 + 2r - 0.3f) \end{pmatrix}.$$

See (11). Consequently,

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial f} \\ \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial f} \end{pmatrix} = \begin{pmatrix} 1/2 - r/5 - f/2 & -r/2 \\ f/5 & -1/10 + r/5 - 3f/50 \end{pmatrix}$$

and

$$A = D\mathbf{F}\begin{pmatrix} 65/103 \\ 90/103 \end{pmatrix} = \begin{pmatrix} -13/206 & -65/206 \\ 18/103 & -27/1030 \end{pmatrix}. \tag{14}$$

The principle of linearization for ODEs is roughly this:

> *Under certain circumstances* the linearized system $\mathbf{y}' = A\mathbf{y}$ captures certain features of the orbit structure associated with the nonlinear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ near the equilibrium point.

I realize there is a good deal of ambiguity in this statement. You should have questions like these:

1. Which features are captured? What exactly can solving the linear system tell me about the phase diagram for the nonlinear system?

2. What does it mean to say "under certain circumstances?" When does linearization fail to give desired information?

Let me try to approach answering these questions using some examples. In our example of the rabbits and foxes a change of notation turns out to be useful. Let us write our system as $\mathbf{r}' = \mathbf{F}(\mathbf{r})$ where $\mathbf{r} = (r, f)^T$. Then the linearized system can be written as $\mathbf{x}' = D\mathbf{F}(\mathbf{r}_*)\mathbf{x}$. We will also write this as $\mathbf{x}' = A\mathbf{x}$ where $A = D\mathbf{F}(\mathbf{r}_*)$ is the constant coefficient derivative (matrix).

When a constant coefficent matrix A is diagonalizable and has **distinct real eigenvalues** $\lambda_1 \neq \lambda_2$, there are recognizable (and important) **straight line solutions**. Let us also assume the matrix A is invertible, so there is a unique equilibrium point of $\mathbf{x}' = A\mathbf{x}$ at the origin. This terminology of "straight line solutions" is my own, but it is a good one. Each straight line solution tells you about three orbits on a line through the equilibrium:

$$\ell_1(t) = e^{\lambda_1 t}\mathbf{v}$$

tells you that two orbits point in toward the equilibrium point along the line in the direction of the eigenvector $\mathbf{v}$ if $\lambda_1 < 0$. These orbits point out if $\lambda_1 > 0$.

Exercise 54 *Why are we not considering the case* $\lambda_1 = 0$*?*

Exercise 55 *Check directly that* $\ell_1(t) = e^{\lambda_1 t}\mathbf{v}$ *is a (straight line) solution of* $\mathbf{x}' = A\mathbf{x}$ *if* $(\lambda_1, \mathbf{v})$ *is an eigenvalue-eigenvector pair for* A*.*

In our case, we have two straight line solutions. The other is

$$\ell_2(t) = e^{\lambda_2 t}\mathbf{w}.$$

In more standard terminology, the lines $\{t\mathbf{v} : t \in \mathbb{R}\}$ and $\{t\mathbf{w} : t \in \mathbb{R}\}$ are called something like **principal manifolds**, and various adjectives are applied according to the sign of the eigenvalue. For example, if $\lambda < 0$, then $\{t\mathbf{v} : t \in \mathbb{R}\}$ is called a **stable manifold**.

Exercise 56 *Draw the phase plane diagram associated with* $\mathbf{x}' = A\mathbf{x}$ *when* A *has the canonical form*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

in the following cases:

(a) $\lambda_1 < \lambda_2 < 0$.

(b) $\lambda_1 < 0 < \lambda_2$.

(c) $0 < \lambda_1 < \lambda_2$.

In the case we are describing, linearization gives a great deal of information about what is happening near the equilibrium point. Take for example the equilibrium point $\mathbf{r}_* = (5, 0)$. The linearized equation at this point is

$$\mathbf{x}' = \begin{pmatrix} -1/2 & -5/2 \\ 0 & 9/10 \end{pmatrix}\mathbf{x}.$$

Notice that it is obvious that the coefficient matrix $A = D\mathbf{r}(5, 0)^T$ here has eigenvalues $\lambda_1 = -1/2$ and $\lambda_2 = 9/10$ with $\mathbf{e}_1$ an eigenvector for λ_1. The eigenvector $\mathbf{w}$ for λ_2 satisfies

$$-\frac{7}{5}w_1 = \frac{5}{2}w_2.$$

Thus, we may take $\mathbf{w} = (25, -14)^T$. This suggests considering the conjugate system

$$\mathbf{y}' = QAQ^1\mathbf{y} = \Lambda\mathbf{y}$$

where

$$Q^{-1} = \begin{pmatrix} 1 & 25 \\ 0 & -14 \end{pmatrix}.$$

Thus, the solutions of the linearized system are given by

$$\mathbf{x}(t) = Q^{-1} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{9t/10} \end{pmatrix} Q\,\mathbf{x}_0.$$

Alternatively, one can write down these solutions directly as a linear combination of the straight line solutions:

$$\mathbf{x}(t) = x_0 e^{-t/2}\mathbf{v} + y_0 e^{9t/10}\mathbf{w}. \tag{15}$$

It will be noticed that there is a **decay direction** (or stable manifold) along the x-axis. Notice how this information translates to information we already know about the nonlinear system $\mathbf{r}' = \mathbf{F}(\mathbf{r})$ near the equilibrium point $\mathbf{r}_* = (5, 0)^T$.

Exercise 57 *Plot the orbits for (15) corresponding to initial points* $(1, 0)$ *and* $(-1, 0)$*. Recall Exercises 44-46, and explain how the behavior for the orbits of the linearized system matches the local behavior of orbits on the r-axis near* $\mathbf{r}_* = (5, 0)^T$ *but not the global behavior.*

Consider the unstable/growth direction for the linearized system along $-\mathbf{w} = (-25, 14)$. The existence of this growth direction tells us there is a special orbit determined by solutions $\mathbf{r} = \mathbf{r}(t)$ starting in the first quadrant and satisfying

$$\lim_{t \searrow -\infty} \mathbf{r}(t) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Every solution in this orbit satisfies also

$$\lim_{t \searrow -\infty} \frac{r_2}{r(t) - 5} = -\frac{14}{25}$$

in accord with the unstable manifold of the linearized system. This special orbit is also called an unstable manifold for the nonlinear system at the equilibrium point $\mathbf{r}_* = (5, 0)^T$. It is also called a **separatrix** because it locally separates certain solutions near the equilibrium point based on their qualitative behavior. In this case, the special orbit separates those solutions with decreasing rabbit population in a neighborhood of the equilibrium (in the first quadrant) from those with increasing rabbit population in a neighborhood of the equilibrium point. Note that all solutions passing near the equilibrium point in the first quadrant have increasing fox population. Compare this to the monotonicity of x_2 in the linearlized system when $x_2 > 0$. This particular separatrix does not have much significance for the global system, especially in forward time. Do you see why? But sometimes, these special orbits play a very significant role in phase plane analysis.

Exercise 58 *How do the slopes of the unstable manifold at* $\mathbb{R}_* = (5,0)^T$ *and that of the vertical nullcline passing through* $(5,0)^T$*?*

Exercise 59 *Draw the phase plane diagram associated with the conjugate system*

$$\mathbf{y}' = \begin{pmatrix} -1/2 & 0 \\ 0 & 9/10 \end{pmatrix} \mathbf{y}.$$

We know the solutions, and consequently the orbits, of the linearized system are obtained by applying the linear transformation associated with Q^{-1} to those of the conjugate system. In particular, the straight line solutions along the eigendirections are easy to understand. This gives us a pretty good picture of the orbit structure for the linearized system. In this case, the equilibrium for $\mathbf{x}' = A\mathbf{x}$ is still called a **hyperbolic equilibrium** or **saddle point**. The same terms apply to the orbit structure of the original nonlinear system near $\mathbf{r}_* = (5,0)^T$. The stable manifold is, in fact, still a straight line along the r-axis, though this property is not always preserved under linearization. The unstable separatrix is not a straight line, but it is a curve meeting the equilibrium point at the same angle as in the linearization. The angle is preserved under linearization.

As a final note about this equilibrium point, the stable manifold along the r-axis may also be considered a separatrix. It is not separating orbits corresponding to meaningful populations, but setting that detail aside, it separates orbits with solutions having increasing f in forward time near the equilibrium point from those having decreasing f in forward time. In this case, the long time global behavior (though not meaningful for this particular model) is very different for solutions on different sides of the separatrix.

7.5 Complex eigenvalues

As mentioned above, the equilibrium point $\mathbf{r}_* = (65/103, 90/103)^T$ is the most important equilibrium point in this system and model. So we return to consider the linearization $\mathbf{x}' = A\mathbf{x}$ with A given by (14). The characteristic equation in this case is

$$\lambda^2 + \frac{130+54}{2060}\lambda + \frac{(13)(27)}{(103)^2(20)} + \frac{(65)(9)}{(103)^2} = 0. \tag{16}$$

The roots may be written as

$$\frac{1}{2}\left[-\frac{46}{(103)(5)} \pm \sqrt{\frac{(46)^2}{(103)^2(25)} - \frac{(9)(13)}{(103)(20)}}\right] = \frac{-46 \mp i\sqrt{58139}}{1030}.$$

A change of notation is again convenient. Let us call the eigenvalue in (16) ζ and write $\zeta = \lambda \mp \mu\, i$. The crucial point is that we have complex eigenvalues.

At this point, if one remembers the general behavior of systems $\mathbf{x}' = A\mathbf{x}$ where the matrix A has complex eigenvalues $\zeta = \lambda \mp i\mu$ with nonzero real part λ, then usually no further work is required. Of course, one needs to know that general behavior before one can remember it. I will show you some ways to know it in the next section, but for now, let me try to tell you what I remember:

1. Complex eigenvalues for $\mathbf{x}' = A\mathbf{x}$ always indicates **rotation** in the sense that orbits cycle around the equilibrium point.
2. If $\lambda < 0$, then solutions spiral in to the equilibrium point, and the equilibrium point is asymptotically stable.
3. If $\lambda > 0$, then solutions spiral out from the equilibrium point, and the equilibrium point is unstable.
4. The direction of the rotation can be read off from the field.

These properties transfer to the local picture for the nonlinear system. In our case, modeling the foxes and rabbits, this is enough. All orbits in the (open) first quadrant spiral in to the stable equilibrium with $r_* = 65/103$ or about 650 rabbits and $f_* = 90/103$ or about 900 foxes.

As to the direction of rotation, one can choose a point of known relation to the equilibrium point and check the field there. For example, the point $(1, 1)^T$ lies roughly up and to the right of the equilibrium point, and not too far away. The field value at this point is

$$\mathbf{F}(1, 1)^T = \begin{pmatrix} -1/10 \\ 7/100 \end{pmatrix}.$$

This direction, it will be noted, is up and to the left indicating a counterclockwise rotation. One comes to the same conclusion by considering the nullclines.

Exercise 60 *Apply the discussion above (with remembered properties) to the canonical form system*

$$\mathbf{y}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}\mathbf{y}.$$

One thing should be noted: If $\lambda = \mathrm{Re}(\zeta) = 0$, that is, one has pure imaginary eigenvalues, then no conclusion may be drawn about the spiraling in or spiraling out of solutions from the linearization. One needs more refined analysis to determine what happens.

Exercise 61 *Solve the canonical form system*

$$\mathbf{y}' = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \mathbf{y}.$$

Hint: write down a single second order equation for y_1, and then ask yourself what functions you know satisfying that second order equation.

7.6 Explicit solutions for systems with complex eigenvalues

We have seen above that one usually does not need to solve a system $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvalues $\zeta = \lambda \mp i\mu$. In order to know the behavior of solutions for such a system, however, one needs to solve it at least once. Here is one approach to solving such a system.

It can always be arranged to have eigenvalue-eigenfunction pairs of the form $\lambda \mp i\mu, \mathbf{v} \pm i\mathbf{w}$) where $\mathbf{v}$ and $\mathbf{w}$ are real vectors.

Exercise 62 *Show that in the case of complex eigenvalues with eigenvalue-eigenfunction pairs $\lambda \mp i\mu, \mathbf{v} \pm i\mathbf{w}$), the set*

$$\{\mathbf{v}, \mathbf{w}\}$$

is a basis for $\mathbb{R}^2$.

Taking a change of basis matrix Q determined by

$$Q^{-1} = \begin{pmatrix} | & | \\ \mathbf{v} & \mathbf{w} \\ | & | \end{pmatrix}$$

where $\mathbf{v}$ and *bw* are the real and imaginary parts of the eigenvectors as described above, the conjugate canonical form is the rotational form

$$\mathbf{y}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{y}.$$

As usual, the solutions of $\mathbf{x}' = A\mathbf{x}$ may be obtained as $\mathbf{x} = Q^{-1}\mathbf{y}$ or, if $\mathbf{y} = \Sigma(t)\mathbf{y}_0$ with inital value $\mathbf{y}_0$ in the conjugate phase space, by

$$\mathbf{x} = Q^{-1}\Sigma(t)Q\mathbf{x}_0$$

where $\mathbf{x}_0 = Q^{-1}\mathbf{y}_0$. As usual, this solution is also $\mathbf{x}(t) = e^{At}\mathbf{x}_0$.

In order to solve the canonical system $\mathbf{y}' = \Lambda\mathbf{y}$ with Λ in rotational canonical form,[4] we proceed as follows: Consider the equation as an equation for $\mathbf{y} : \mathbb{R} \to \mathbb{C}^2$ instead of $\mathbf{y} : \mathbb{R} \to \mathbb{R}^2$. Over $\mathbb{C}$ the matrix Λ is diagonalizable. In fact, the characteristic equation is

$$(\zeta - \lambda)^2 + \mu^2 = 0,$$

so the eigenvalues are the same $\zeta = \lambda \mp i\mu$, but the eigenvectors are easily seen to be $\mathbf{e}_1 \pm i\mathbf{e}_2$. Thus, there are straight line solutions $\ell_1 : \mathbb{R} \to bbc^2$ given by

$$\ell_1(t) = e^{(\lambda - i\mu)t}(\mathbf{e}_1 + i\mathbf{e}_2)$$

and $\ell_2 : \mathbb{R} \to bbc^2$ given by

$$\ell_2(t) = e^{(\lambda - i\mu)t}(\mathbf{e}_1 + i\mathbf{e}_2).$$

We could work with these directly, but (just to make sure we understand the details) let's diagonalize. Remember that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is still a basis for $\mathbb{C}^2$ over $\mathbb{C}$, so we can take $\{\mathbf{e}_1 + i\mathbf{e}_2, \mathbf{e}_1 - i\mathbf{e}_2\}$ as a second basis to determine a change of basis matrix R by

$$R^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Notice that the columns of R^{-1} are the two complex vectors in the second basis (written in coordinates with respect to the standard basis) as usual. A second conjugate system is then given by

$$\mathbf{z}' = M\mathbf{z}$$

where $M = R\Lambda R^{-1}$ is the complex diagonal matrix

$$M = \begin{pmatrix} \lambda - i\mu & 0 \\ 0 & \lambda + i\mu \end{pmatrix}.$$

The system $\mathbf{z}' = M\mathbf{z}$ for $\mathbf{z} : \mathbb{R} \to \mathbb{C}^2$ decouples and has general solution

$$\mathbf{z}(t) = \begin{pmatrix} \zeta_1 e^{(\lambda - i\mu)t} \\ \zeta_2 e^{(\lambda + i\mu)t} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} e^{-i\mu t} & 0 \\ 0 & e^{i\mu t} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

From this solution, we can read off the general solution for $\mathbf{y}' = \Lambda\mathbf{y}$:

$$\mathbf{y}(t) = R^{-1}\mathbf{z}(t) = e^{\lambda t} R^{-1} \begin{pmatrix} e^{-i\mu t} & 0 \\ 0 & e^{i\mu t} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = e^{\lambda t} R^{-1} \begin{pmatrix} e^{-i\mu t} & 0 \\ 0 & e^{i\mu t} \end{pmatrix} R\mathbf{y}_0.$$

[4]You may wish to have a look at Exercise 63 below at this point to help you better anticipate what should come out of the solution to follow.

We need to compute the inverse of the change of basis matrix R^{-1} to get R:

$$R = -\frac{1}{2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Thus,

$$R^{-1}\begin{pmatrix} e^{-i\mu t} & 0 \\ 0 & e^{i\mu t} \end{pmatrix}R = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{-i\mu t} & -ie^{-i\mu t} \\ e^{i\mu t} & ie^{i\mu t} \end{pmatrix} = \begin{pmatrix} \cos\mu t & -\sin\mu t \\ \sin\mu t & \cos\mu t \end{pmatrix}.$$

This is the key calculation. It tells us that

$$\mathbf{y}(t) = e^{\lambda t}\begin{pmatrix} \cos\mu t & -\sin\mu t \\ \sin\mu t & \cos\mu t \end{pmatrix}\mathbf{y}_0.$$

This gives all solutions of $\mathbf{y}' = \Lambda\mathbf{y}$ for $\mathbf{y} : \mathbb{R} \to \mathbb{C}^2$ including any real ones, if there are any. Indeed, the matrix Λ is a real matrix, so the specification $\mathbf{y}' = \Lambda\mathbf{y}$ says that if $\mathbf{y}$ is a real vector, then $\mathbf{y}$ should "move" in a real direction. It is very natural to have real solutions. In particular, taking $\mathbf{y}_0$ to be $\mathbf{e}_1$ and $\mathbf{e}_2$, we obtain the basis of real solutions $\{\mathbf{y}_1, \mathbf{y}_2\}$ given by

$$\mathbf{y}_1(t) = e^{\lambda t}\begin{pmatrix} \cos\mu t \\ \sin\mu t \end{pmatrix} \qquad \text{and} \qquad \mathbf{y}_2(t) = e^{\lambda t}\begin{pmatrix} -\sin\mu t \\ \cos\mu t \end{pmatrix}.$$

These two solutions span all solutions $\mathbf{y} : \mathbb{R} \to \mathbb{C}^2$ over $\mathbb{C}$, and they also span all real solutions over $\mathbb{R}$.

Exercise 63 *Check directly that the system* $\mathbf{y}' = \Lambda\mathbf{y}$ *with*

$$\Lambda = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$$

has two linearly independent solutions

$$\mathbf{y}_1(t) = e^{\lambda t}\begin{pmatrix} \cos\mu t \\ \sin\mu t \end{pmatrix} \qquad \text{and} \qquad \mathbf{y}_2(t) = e^{\lambda t}\begin{pmatrix} -\sin\mu t \\ \cos\mu t \end{pmatrix},$$

so that the general solution may be written as a linear combination $\mathbf{y} = a\mathbf{y}_1 + b\mathbf{y}_2$.

Write the general solution in the form

$$\mathbf{y}(t) = \Sigma(t)\mathbf{y}_0$$

where Σ *is a matrix valued function of* t. *This matrix* Σ *is* $e^{\Lambda t}$.

Finally, the explicit general solution for $\mathbf{x}' = A\mathbf{x}$ where A has complex eigenvalues $\zeta = \lambda \mp i\mu$ is

$$\mathbf{x}(t) = e^{\lambda t} Q^{-1} \begin{pmatrix} \cos \mu t & -\sin \mu t \\ \sin \mu t & \cos \mu t \end{pmatrix} Q\mathbf{x}_0.$$

Exercise 64 *Draw the matrix/mapping diagrams for the first and second conjugate systems associated with $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvalues $\zeta = \lambda \mp i\mu$.*

8 Summary for linear constant coefficient systems

Every linear constant coefficient system $\mathbf{x}' = A\mathbf{x}$ is conjugate to a canonical system $\mathbf{y}' = \Lambda\mathbf{y}$. In the real 2×2 case there are essentially three canonical forms to consider: diagonal, Jordan, and rotational. Solving a system $\mathbf{y}' = \Lambda\mathbf{y}$ when Λ is a real diagonal diagonal matrix is easy

$$\mathbf{y}(t) = ae^{\lambda_1 t}\mathbf{e}_1 + be^{\lambda_2 t}\mathbf{e}_2.$$

Solving a diagonalizable system $\mathbf{x}' = A\mathbf{x}$ is just as easy using straight line solutions:

$$\mathbf{x}(t) = x_0 e^{\lambda_1 t}\mathbf{v} + y_0 e^{\lambda_2 t}\mathbf{w}.$$

The Jordan form system partially decouples and is also not too difficult to solve. The rotational form system is relatively easy to solve if you remember the form of the solutions, though the conjugate system you obtained from linearization probably has a relatively complicated solution. Fortunately, one usually does not need to solve these systems, but it is enough to remember the basic behavior of solutions for the ODE $\mathbf{x}' = A\mathbf{x}$ when A has complex eigenvalues.

9 Existence and Uniqueness

A simple version of the general existence and uniqueness theorem is the following:

Theorem 13 *If $\mathbf{F} \in C^0(\mathbb{R}^n \times (a,b) \to \mathbb{R}^n)$ and for each fixed t the function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$\mathbf{v}(\mathbf{x}) = \mathbf{F}(\mathbf{x}, t)$$

satisfies $\mathbf{v} \in Lip_{loc}(\mathbb{R}^n)$, then for any $(\mathbf{x}_0, t_0) \in \mathbb{R}^n \times (a,b)$, there is some $\delta > 0$ for which the IVP

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}, t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution $\mathbf{x} \in C^1((t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n)$.

As usual, this should be contrasted to the **linear existence and uniqueness** theorem along the following lines:

Theorem 14 *If $A = A(t)$ is a matrix valued function depending continuously on $t \in (a, b)$, more precisely, $A \in C^0((a, b) \to M^{n\times n})$ where $M^{n\times n}$ denotes the ring of $n \times n$ matrices (whose entries may be taken to be either real or complex), and $\mathbf{b} \in C^0((a, b) \to \mathbb{R}^n$, then for any $t_0 \in (a, b)$ and any $\mathbf{x}_0 \in \mathbb{R}^n$ the IVP*

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution $\mathbf{x} \in C^1((a, b) \to \mathbb{R}^n)$. In short, if the coefficients are continuous, then one can count on existence and uniqueness.

10 Higher Order Equations

The discussion above may have left you asking the following questions:

1. What about second order linear equations?
2. Where do Laplace transforms fit in with all this?
3. What about higher order equations?

Indeed second order equations (not just linear ones) can sometimes be important and are worth thinking about on their own, especially in for oscillators and also in physical systems modeled by Newton's second law

$$m\mathbf{x}'' = \mathbf{F}$$

which is a second order ODE with $\mathbf{F}$ depending typically on $\mathbf{x}$, $\mathbf{x}'$ and t. Thus, one-dimensional motion, when $\mathbf{x} = x$ is real valued, is most naturally treated in terms of a single second order ODE. The special case of a one-dimensional linear **oscillator** is modeled with an equation of the form

$$mx'' = -px' - qx + f$$

where p and q are positive constants (or possibly functions of t and f is a function of t called the inhomogeneiety or forcing. The equation(s) modeling RLC circuits also fall into this category of oscillators, and these can be more easily treated (at least in some respects) as single second order equations.

Generally, speaking any ODE of order k has the form

$$G(\mathbf{x}^{(k)}, \mathbf{x}^{(k-1)}, \ldots, \mathbf{x}', \mathbf{x}, t) = 0$$

where G is a function of many (the appropriate number) of variables. That number would be $nk + 2$ if $\mathbf{x}$ takes values in $\mathbb{R}^n$. It is usual to restrict attention to **regular equations** which may be written in the form

$$\mathbf{x}^{(k)} = f(\mathbf{x}^{(k-1)}, \ldots, \mathbf{x}', \mathbf{x}, t). \tag{17}$$

The first key observation is the following

Every single ODE (of any order) is equivalent to a first order system.

In fact, every higher order system like (17) is also equivalent to a first order system. Thus, if our discussion of the system $\mathbf{x}' = \mathbf{F}(\mathbf{x}, t)$ were carried to completion, we would then have obtained any information that can be obtained about a single ODE. To see the equivalence, start with a single regular k-th order ODE

$$y^{(k)} = f(y^{(k-1)}, \ldots, y', y, t), \tag{18}$$

and set $x_1 = y$. Then we consider the system of equations:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{k-1}' = x_k \\ x_k' = f(x_k, x_{k-1}, \ldots, x_2, x_1, t). \end{cases}$$

Notice that if we can find a solution $\mathbf{x} \in C^1((a, b) \to \mathbb{R}^k)$ of this system, then letting $x = x_1$ be the first coordinate, we find for $j = 1, 2, \ldots, k$

$$x^{(j)} = \frac{d^j x}{dt^j} = x_{j+1}.$$

Thus, $x \in C^k(a, b)$ and the last equation in the system says that x is a solution of the single ODE (18).

If we have a solution $y \in C^k(a, b)$ of the original equation, then $\mathbf{x} : (a, b) \to \mathbb{R}^k$ by

$$\mathbf{x}(t) = (y(t), y'(t), \ldots, y^{(k-1)}(t))$$

gives a solution of the system with $\mathbf{x} \in C^1((a, b) \to \mathbb{R}^k)$. Thus, the system is equivalent to the single equation.

This equivalence also tells us the natural initial values/conditions for a single equation: Since the system requires a starting point

$$\mathbf{x}_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_k(t_0) \end{pmatrix} = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ \vdots \\ y^{(k-1)}(t_0) \end{pmatrix},$$

the natural initial conditions for the single equation are of the form

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{k-1}(t_0) = y_0^{(k-1)}$$

where $y_0, y_0', \ldots, y_0^{(k-1)}$ are some given k real numbers (or complex numbers).

Exercise 65 *State existence and uniqueness theorems for single nonlinear and linear ODEs of order k.*

Having said all this, it is often convenient to think about single higher order equations directly in some framework of single higher order equations rather than in the (equivalent) framework of systems. In particular, linear theory as applied to single higher order linear equations is very familiar and convenient. One has

$$Lu = f$$

where L is a k-th order linear operator. There is an associated homogeneous equation and the search for particular solutions. The kernel of L is, using the equivalent system, seen to be a vector space of dimension k.

Exercise 66 *Given a linear operator* $L : C^k(a,b) \to C^0(a,b)$ *by*

$$Lu = \sum_{j=0}^{k} a_j(t) \frac{d^j u}{dx^j}$$

where $a_k \equiv 1$*, use the equivalent system and the linear existence and uniqueness theorem to show*

$$\ker(L) = \{u \in C^k(a,b) : Lu = 0\}$$

is a k-dimensional subspace. Hint: Initial values $\mathbf{e}_j$ *for* $j = 1, 2, \ldots, k$*.*

Looking at the form of the inhomogeneity f and guessing the form of a particular solution, with appropriate constants to be determined by plugging the guess into the operator, has been dignified as the **method of undetermined coefficients** and has associated with it a fairly well-defined set of techniques for guessing the forms.

A kind of algebraic approach to solving the homogeneous equation and finding a particular solution solving a specific initial value problem is called **the method of Laplace transforms**. Aside from being a time-saving algebraic technique in some instances, the main contribution of the Laplace transform method from the engineering point of view is that it offers a way to model **impulse forcing** which is not easily modeled in the framework of conventional forcing functions.

These are topics which you may wish to review, but are being left out of this course as somewhat specialized techniques not directly related to an overview of ODEs.