Chapter 1

Cauchy's Theorems

We may take complex analysis to be the study of functions $f : U \to \mathbb{C}$ which are **complex differentiable**. The set U here is required to be **open** in $\mathbb{C}$, that is, for each $z_0 \in U$, there is some $r > 0$ so that

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset U \tag{1.1}$$

where

$$|z - z_0| = \sqrt{(z - z_0)\overline{(z - z_0)}} = \sqrt{[\mathrm{Re}(z - z_0)]^2 + [\mathrm{Im}(z - z_0)]^2}.$$

True or False: $[\mathrm{Re}\, z]^2 = \mathrm{Re}\, z^2$?
True or False: $(\mathrm{Re}\, z)(\mathrm{Re}\, \overline{z}) = \mathrm{Re}(z\overline{z})$?
Complex differentiable means that for each $z_0 \in U$ the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We may expand what we can show later, but for now we will assume the function $f' : U \to \mathbb{C}$ is also continuous.

The main objective of this chapter is to present two theorems about such a complex differentiable function. In each case there is also an open set Ω with $\Omega \subset\subset U$. This condition is read Ω is **compactly contained** in U, meaning in this case Ω is bounded and $\overline{\Omega} \subset U$. For the theorems the set Ω is required to satisfy other more stringent conditions as well that I won't describe right now. The open set U on the other hand may be quite arbitrary; the main restriction on U is that it contain the closure of Ω.

Explaining the most general possibilities for Ω can be very tricky. A prototypical domain Ω is given by a disk

$$\Omega = B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

where $r > 0$. Remember we are requiring $\Omega \subset\subset U$, so this can be a little different than the consideration of a ball in (1.1) where I was explaining what it means for U to be open. Here we require

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset U. \tag{1.2}$$

Exercise 1.1. Show by example that (1.2) can be different from (1.1).

Roughly speaking we need the boundary of the set Ω under consideration to be "reasonably well behaved," that is a "reasonably nice" simple closed curve. We also want Ω to be a bounded set, so we are not particularly interested in

$$\Omega = \{z \in \mathbb{C} : |z - z_0| > r\} = \mathbb{C}\backslash\overline{B_r(z_0)},$$

a set with closure contained in $U = \mathbb{C}\backslash\overline{B_{r/2}(z_0)}$. Remember it is part of the definition of $\Omega \subset\subset U$ (Ω is compactly contained in U) that Ω be also a bounded set. We are also not interested in an "annular" set like

$$\Omega = B_r(z_0)\backslash\overline{B_{r/2}(z_0)}$$

which does satisfy $\Omega \subset\subset U = \mathbb{C}$ for example. There are a couple reasons for this. Primarily we need some notion of **orientation** for $\Gamma = \partial\Omega$ so we can integrate on $\partial\Omega$ in a directional way that is compatible with the entire domain Ω. . The set $\Omega = B_r(z_0)\backslash\overline{B_{r/2}(z_0)}$ has two boundary components. The usual way to orient such a domain is to either associate with it the outward normal to $\partial\Omega$ or the inward normal. Note that this gives two "orientations" for the domain Ω itself. The problem is that once such an orientation for Ω is chosen, an orientation on the boundary components/curves is also induced and one ends up with one circle having a clockwise orientation and the other a counter-clockwise orientation. Roughly speaking the theorems requires a single unified orientation for the entire boundary.

Setting aside the details of how orientation for boundary curves might work and the most general domains we might consider, I'll mention that for $r > 0$ and $z_0 \in \mathbb{C}$

$$\Omega = \{z \in \mathbb{C} : |\operatorname{Re} z - \operatorname{Re} z_0| < r \text{ and } |\operatorname{Im} z - \operatorname{Im} z_0| < r\} \tag{1.3}$$

is another acceptable kind of set for the theorems, and I'll state the theorems. In the mean time you can contemplate a family of open sets Ω general enough to include the square ball (1.3) but not so general that the theorems don't hold or don't make sense.

Theorem 1. (Cauchy's integral theorem) If $f : U \to \mathbb{C}$ is complex differentiable on an open set $U \subset \mathbb{C}$ and there is a bounded open set $\Omega \subset\subset U$ with boundary a "reasonably nice" clockwise oriented simple closed curve $\Gamma = \partial\Omega$, then

$$\int_\Gamma f = 0.$$

Theorem 2. (Cauchy's integral formula) If $f : U \to \mathbb{C}$ is complex differentiable on an open set $U \subset \mathbb{C}$ and there is a bounded open set $\Omega \subset\subset U$ with boundary a "reasonably nice" clockwise oriented simple closed curve $\Gamma = \partial\Omega$, then

$$f(z) = \frac{1}{2\pi i} \int_{w\in\Gamma} \frac{f(w)}{w - z}$$

for all $z \in \Omega$.

You can also start to think about what parts of the statements of these theorems need more explaining. In the mean time, I'll start trying to explain why they are true.

1.1 $\mathbb{R}^2$ and $\mathbb{C}$

There is a natural identification of an open set $U \subset \mathbb{C}$ with an open subset of $\mathbb{R}^2$. This identification is somewhat important for the discussion below, so I will now introduce some standardized notation to go along with it. The basic identification is determined by the function $\check{\imath} : \mathbb{C} \to \mathbb{R}^2$ by

$$\check{\imath}(z) = (\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^2.$$

This function may be considered a kind of injection or inclusion. In many ways the sets $\mathbb{R}^2$ and $\mathbb{C}$ are very much alike. In calculus when one is working with real spaces like $\mathbb{R}^2$, it is natural to have the basic notation of points and subsets used in the explanation of definitions, concepts, assertions, etc., and in that context extensions and other constructions are considered "exotic" or "other" and merit alternative notation. In this discussion the set $\mathbb{C}$ plays

the nominal central role, so the "check" inclusion $\check{\iota}$ is introductng a notation designating the real space $\mathbb{R}^2$ as "other."

For example, I will denote by $\check{U}$ the set

$$\check{U} = \{(x, y) : x + iy \in U\} \subset \mathbb{R}^2$$

where I am assuming $x = \mathrm{Re}(x + iy)$ and $y = \mathrm{Im}(x + iy)$.

Exercise 1.2. If $x = \mathrm{Re}(x+iy)$ for $x+iy \in \mathbb{C}$, is it true that $y = \mathrm{Im}(x+iy)$?

Similarly, I will write

$$\check{B}_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{p}| < r\} = \check{\iota}(B_r(z_0))$$

when $\mathbf{p} = \check{\iota}(z_0)$. Notice that sometimes there are "checks" to alert you (and me) to the fact that real spaces are under consideration (in contrast to the complex space $\mathbb{C}$); other times no notation appears, and hopefully no notation is necessary. I could write $\check{z}_0 = \check{\iota}(z_0)$, but instead I have used simply $\mathbf{p}$.

The function $f : U \to \mathbb{C}$ may be identified with a function $\check{f} : \check{U} \to \mathbb{R}^2$ with

$$\check{f}(x, y) = (\mathrm{Re}\, f(x + iy), \mathrm{Im}\, f(x + iy)).$$

Though we will not often use the notation $\check{f}$, this vector valued function of two variables has two important interpretations, each of which usually receives its own notation. The component functions of $\check{f}$ in particular will be referred to as

$$u(x, y) = \mathrm{Re}\, f(x + iy) \qquad \text{and} \qquad v(x, y) = \mathrm{Im}\, f(x + iy).$$

1.2 Calculus III for complex analysis

Note $u, v : \check{U} \to \mathbb{R}$, and these functions may be used to define a (real) vector field $\mathbf{F} : \check{U} \to \mathbb{R}^2$ on $\check{U}$ by

$$\mathbf{F}(x, y) = (u(x, y), v(x, y)).$$

This is the same function $\check{f}$ to which we will not often refer. Given the vector field $\mathbf{F}$, the functions u and v are other names for the coordinate functions in

$\mathbf{F} = (F_1, F_2)$. If u and v (or F_1 and F_2) are **continuously differentiable**, a condition meaning the partial derivatives

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \text{ and } \frac{\partial v}{\partial y}$$

exist and are continuous, then we may use **Green's theorem**:

Theorem 3. (Green's theorem) If $\check{\Gamma} = \partial\check{\Omega}$ is a piecewise regular simple closed curve in $\mathbb{R}^2$, then

$$\int_{\check{\Gamma}} \mathbf{F} \cdot \vec{T} = \int_{\check{\Omega}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right). \tag{1.4}$$

There are perhaps a few things to recall or review or note about (the statement of) Green's theorem.

First a **piecewise regular** curve admits a parameterization

$$\check{\alpha} \in C^0([a,b] \to \mathbb{R}^2)$$

on a partitioned interval

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

with

$$\check{\alpha}'_j = \check{\alpha}\big|_{[t_j, t_{j+1}]} \in C^1([t_j, t_{j+1}] \to \mathbb{R}^2) \qquad \text{for} \qquad j = 0, 1, 2, \ldots, n-1$$

and

$$\check{\alpha}'_j = \begin{cases} \check{\alpha}'(t_j^+), & t = t_j \\ \check{\alpha}'(t), & t \in (t_j, t_{j+1}) \\ \check{\alpha}'(t_{j+1}^-), & t = t_{j+1} \end{cases} \qquad \text{for} \qquad j = 0, 1, 2, \ldots, n-1$$

satisfying $\check{\alpha}'_j(t) \neq \mathbf{0}$ for all t and j, and

$$\check{\alpha}'_{j+1}(t_{j+1}) \cdot \check{\alpha}'_j(t_{j+1}) \neq -|\check{\alpha}'_{j+1}(t_{j+1})| \, |\check{\alpha}'_j(t_{j+1})| \qquad \text{for} \qquad j = 0, 1, 2, \ldots, n-2.$$

If $\check{\Gamma}$ is required to be **closed** (as in a closed curve) as is the case in Green's theorem, then $\check{\alpha}(b) = \check{\alpha}(a)$ and

$$\check{\alpha}'(a) \cdot \check{\alpha}'(b) \neq -|\check{\alpha}'(a)| \, |\check{\alpha}'(b)|$$

as well.

Simple means the curve does not "cross itself." This condition can be expressed in terms of the parameterization by the following condition:

If $\check{\alpha}(t_1) = \check{\alpha}(t_2)$ for $t_1 \neq t_2$, then $t_1, t_2 \in \{a, b\}$.

General integration is defined on such curves $\check{\Gamma} \subset \mathbb{R}^2$ aside from orientation as follows:

Definition 1. (real integral on a curve) Given $g \in C^0(\check{\Gamma} \to \mathbb{C})$ and setting $\check{\Gamma}_j = \check{\alpha}[t_j, t_{j+1}]$ for $j = 0, 1, 2, \ldots, n-1$,

$$\int_{\check{\Gamma}_j} g = \lim_{\|\mathcal{P}\| \to 0} \sum_k g(p_k^*) \operatorname{length}(\check{\Gamma}_{jk})$$

where $\mathcal{P} = \{\check{\Gamma}_{jk}\}_k$ is a partition of

$$\check{\Gamma}_j = \bigcup_k \check{\Gamma}_{jk}$$

and $p_k^* \in \check{\Gamma}_{jk}$ is an evaluation point;

$$\int_{\check{\Gamma}} g = \sum_{j=0}^{n-1} \int_{\check{\Gamma}_j} g.$$

We have defined here a real integral of a complex valued function g; the real integral of a real valued function $g : \check{\Gamma} \to \mathbb{R}$ is a special case.

In Green's theorem and in the integral

$$\int_{\check{\Gamma}} \mathbf{F} \cdot \vec{T}$$

in particular, the function $g : \check{\Gamma} \to \mathbb{R}$ is given by

$$g(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \cdot \frac{\check{\alpha}'(t)}{|\check{\alpha'}(t)|}$$

where $\check{\alpha}(t) = \mathbf{x} \in \check{\Gamma}$. Note this function g is well-defined except at the corner points $\{\check{\alpha}(t_j)\}_{j=0}^{n-1}$, and since each restriction

$$g_j = g\Big|_{\check{\alpha}(t_j, t_{j+1})} \in C^0(\check{\alpha}(t_j, t_{j+1})) \qquad \text{for} \qquad j = 0, 1, 2, \ldots, n-1$$

extends uniquely to $\check{\alpha}[t_j, t + j + 1]$, the finitely many corner points will not effect the value of the integral.

Definition 2. (real integral on a full dimension set $\Omega \subset \mathbb{R}^2$) Given $g \in C^0(\overline{\Omega} \to \mathbb{C})$,

$$\int_\Omega g = \lim_{\|\mathcal{P}\| \to 0} \sum_j g(p_j^*) \operatorname{area}(\Omega_j)$$

where $\mathcal{P} = \{\Omega_j\}_j$ is a partition of

$$\Omega = \bigcup_j \Omega_j$$

and $p_j^* \in \Omega_j$ is an evaluation point.

Notice I might have (and maybe should have) used $\check{\Omega} = \check{\imath}(\Omega)$ to denote the domain in $\mathbb{R}^2$ in this definition. There are a couple reasons I didn't. For one I needed to mention the closure of the domain, and this would leave me with $\overline{\check{\Omega}}$ or $\overline{\check{\imath}(\Omega)}$ which I deemed typographically cumbersome. Second and more importantly there will be no such thing as a complex integral on a full dimension set $\Omega \subset \mathbb{C}$. Complex integration is only for curves. Thus, there is no need to draw a distinction between real integrals on areas (areal, full dimension regions) in $\mathbb{R}^2$ (or more generally full dimension sets in $\mathbb{R}^n$) with any other kind of integral.

Finally, the curve in Green's theorem is required to **oriented in the counterclockwise direction**. This can be taken to mean the vector

$$-(\check{\alpha}')^\perp = (\check{\alpha}_2', -\check{\alpha}_1')$$

point out of $\Omega = \check{\Omega}$ along $\check{\Gamma} = \partial\check{\Omega}$. Consequently

$$\vec{N} = -\vec{T}^\perp$$

point out of Ω as well.

1.3 Formulas for real integration

If $\check{A}$ is a section of a curve $\check{\Gamma}$ providing a domain of integration for the function $g \in C^0(\check{\Gamma} \to \mathbb{C})$ and $\check{\alpha} \in C^1(I \to \mathbb{R}^2)$ is a parameterization of $\check{A}$ on an interval $I = [a_1, a_2]$, then

$$\int_{\check{A}} g = \int_{a_1}^{a_2} g \circ \check{\alpha}(t)\ |\check{\alpha}'(t)|\, dt.$$

If

$$\Omega = \{(x, y) \in \mathbb{R}^2 : a_1 < x < a_2,\ h_1(x) < y < h_2(x)\}$$

provides a domain of integration for $g \in C^0(\overline{\Omega} \to \mathbb{C})$, then

$$\int_\Omega g = \int_{a_1}^{a_2} \int_{h_1(x)}^{h_2(x)} g(x, y)\, dy\, dx.$$

This is called an **iterated integral** and this is one version of Fubini's theorem.

Exercise 1.3. Formulate another version of Fubini's theorem for a domain of integration $\Omega \subset \mathbb{R}^2$ which is a union of horizontal line segments.

We will not use the formula for integration on a full dimensional region as much as we will use the formula (and others like it) for integration on a curve.

1.4 Complex integrals

Like a complex derivative borrows from the familiar form of a real derivative, the form of a complex integral borrows from the form of a real integral, but due to the incorporation of complex multiplication, it is a very different thing. Here, returning to the complex plane $\mathbb{C}$ we may dispense with the "check" notation.

Definition 3. (complex integral) Given an oriented curve $\Gamma \subset \mathbb{C}$ and $f \in C^0(\Gamma \to \mathbb{C})$

$$\int_\Gamma f = \lim_{\|\mathcal{P}\| \to 0} \sum_j f(z_j^*)(z_{j+1} - z_j).$$

Most of what is happening here, though it is a little complicated, is entirely analogous to the case of real integration. The only real difference[1] is that the **complex increment** $z_{j+1} - z_j$ like the complex increment $w - z$ in the difference quotient

$$\frac{f(w) - f(z)}{w - z}$$

[1] Pun intended.

interacts in a new and interesting way with the function value(s) $f(z_j^*)$. The partition $\mathcal{P}$ is a collection of pieces Γ_j of the curve Γ. (No "checks" needed.) The pieces intersect in endpoints $\Gamma_{j+1} \cap \Gamma_j = \{z_j\}$ and

$$\Gamma = \bigcup_k \Gamma_j.$$

The index k can be taken to run from 0 to $n-1$ where $n \in \mathbb{N}$ is some finite number. Though in the limit there are many "arbitrary" participants, the pieces Γ_j and the number n of those pieces among them. Also, the evaluation points z_j^* are arbitrary aside from the requirement $z_j^* \in \Gamma_j$. The norm of the partition is

$$\|\mathcal{P}\| = \max_{0 \le j < n} \operatorname{diam} \Gamma_j = \max_{0 \le j < n} \sup\{|w - z| : w, z \in \Gamma_j\}.$$

With this definition we have a handful of different kinds of integrals.

The integral of $g : I \to \mathbb{R}$ where $I = [a_1, a_2]$ is an interval of real numbers.

$$\int_{a_1}^{a_2} g(t)\, dt = \lim_{\|\mathcal{P}\| \to 0} \sum_j g(t_j^*)(t_{j+1} - t_j).$$

The partition here is usually expressed as a collection of points

$$a_1 = t_0 < t_1 < t_2 < \cdots < t_n = a_2.$$

As pieces of the interval I, the partition $\mathcal{P}$ would consist of subintervals $[t_j, t_{j+1}]$ for $j = 0, 1, 2, \ldots, n-1$. This same definition works for the real integral of the complex valued function $g : I \to \mathbb{C}$.

There are many formulas for this kind of integral with specific functions g for the integrand covered in Calculus and especially Calculus II (which is at least traditionally mainly devoted to "techniques of integration." Nevertheless, even after one finishes Calculus II there are still many functions $g \in C^0(I)$ for which the integral is well defined, but there is no other notion of that integral aside from the limit definition given above.

A generalization of this, is the real integral on a curve $\check{\Gamma}$. We've mentioned/described the complication of what it means for Γ to be piecewise C^1. Ignoring that aspect, we can simply write the real integral of $g : \Gamma \to \mathbb{C}$ as

$$\int_{\check{\Gamma}} g = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=0}^{n-1} g(\mathbf{x}_j^*) \operatorname{length}(\check{\Gamma}_j)$$

where $\mathcal{P} = \{\check{\Gamma}_j\}_{j=0}^{n-1}$ is a partition of $\check{\Gamma}$ and $\mathbf{x}_j^* \in \check{\Gamma}_j$. Here we retain the "check" to emphasize that these are real integrals on a curve in $\mathbb{R}^2$, though the curve could indeed be a curve in $\mathbb{R}^n$ with precisely the same definition.

The "check" is to draw contrast with the complex integral

$$\int_\Gamma f = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=0}^{n-1} f(z_j^*)(z_{j+1} - z_j)$$

which is the main object of attention in this chapter and especially this section. In each case, one can say the integral is a limit of a Riemann sum as the norm of a partition tends to zero. The last integral differs from the others however in that the increment is a complex number rather than a length. This makes all the difference.

We could also repeat the definition(s) of integrals on higher dimensional sets, at least full dimension subsets of $\mathbb{R}^n$ for $n \geq 2$. Instead, let's observe two relations between complex integrals and real path integrals.

In the case $\check{\Gamma}$ is a C^1 curve with parameterization $\check{\alpha} : I \to \mathbb{R}^2$, we gave a formula for the real integral of a function g over $\check{\Gamma}$ in terms of a real integral over the interval I. That formula was

$$\int_{\check{\Gamma}} g = \int_{a_1}^{a_2} g \circ \check{\alpha}(t)\, |\check{\alpha}'(t)|\, dt. \tag{1.5}$$

Notice the integral expression

$$\int_{a_1}^{a_2} g \circ \check{\alpha}(t)\, \check{\alpha}'(t)\, dt$$

is nonsense. One must have the Euclidean length of the vector $\check{\alpha}'(t)$ in there or else the product $g \circ \check{\alpha}(t)\, \check{\alpha}'(t)$ is a scaling of the vector $\check{\alpha}'(t)$, which is again a vector. If g is real valued, then this vector is again in $\mathbb{R}^2$, and if g is complex valued, then the product is in $\mathbb{C}^2$ which is actually getting a little strange because we started thinking of $\check{\alpha}'(t)$ as a vector in $\mathbb{R}^2$, so scaling by complex numbers is not really a thing there. Of course, it might be observed that $\mathbb{R}^2$ can be "realized" as a kind of interesting subset (though not subspace) of $\mathbb{C}^2$, so perhaps some sense could be made of this vector. But the main point is, we have not defined any kind of integral of vector valued functions.

Actually, we could define an integral of a vector valued function "componentwise," and this is essentially what we have done for real integrals of

complex valued functions where the "components" are the real and imaginary parts. But we haven't done this for $\mathbb{R}^2$ or $\mathbb{C}^2$. That would be a different thing. As I say we could do this, but we haven't. Let's ignore this kind of integral for the moment.

The first relation between complex integrals and real integrals is somewhat similar to (1.5). For $f \in C^0(\Gamma \to \mathbb{C})$ where Γ is a C^1 curve in $\mathbb{C}$ with parameterization $\alpha : I \to \mathbb{C}$ on an interval $I = [a_1, a_2]$, the complex integral of f over Γ can be expressed as a real integral over I:

$$\int_\Gamma f = \int_I f \circ \alpha \; \alpha' = \int_{a_1}^{a_2} f \circ \alpha(t) \, \alpha'(t) \, dt. \tag{1.6}$$

In (1.5) for a real path integral the orientation of $\check{\Gamma}$ doesn't really matter. In this formula, however, the curve Γ must be oriented in the sense of having a direction from one endpoint z_1 at the beginning and one endpoint z_2 at the end. The parameterization $\alpha : I \to \mathbb{C}$ in formula (1.6) must be a parameterization respecting the orientation of Γ with $\alpha(a_1) = z_1$ and $\alpha(a_2) = z_2$. In general, the product $f \circ \alpha \; \alpha'$ will be a complex number, so the integral on the right of (1.6) is a real integral of a complex valued function.

The second relation is an inequality:

$$\left| \int_\Gamma f \right| \leq \int_{\check{\Gamma}} |\check{f}|. \tag{1.7}$$

In this form we have used the seldom mentioned "realification" $\check{f} : \check{\Gamma} \to \mathbb{R}^2$ of f given by

$$\check{f}(x, y) = (\operatorname{Re} f(x + iy), \operatorname{Im} f(x + iy)) = (u(x, y), v(x, y)).$$

Let us rewrite (1.7) in the case where Γ admits a C^1 parameterization $\alpha : I \to \mathbb{C}$ with $I = [a_1, a_2]$. In this case we have

$$\left| \int_\Gamma f \right| \leq \int_{a_1}^{a_2} |f \circ \alpha(t)| \; |\alpha'(t)| \, dt.$$

Notice that in this form the functions $|f \circ \alpha| : I \to \mathbb{R}$ and $|\alpha'| : I \to \mathbb{R}$, so the product appearing in the integrand on the right is a real valued function of one real variable on an interval I. Consequently, this is a form that avoids all vexing "checks." Also, orientation doesn't matter in this formula.

One final note about (1.7): If one happens to have a bound $|f(z)| \leq M$ for all $z \in \Gamma$, then (1.7) implies a nice "check" free inequality:

$$\left| \int_\Gamma f \right| \leq M \operatorname{length}(\Gamma),$$

where $\operatorname{length}(\Gamma)$ may be defined as

$$\lim_{\|\mathcal{P}\| \to 0} \sum_j |z_{j+1} - z_j|.$$

1.5 Cauchy's integral theorem

Let Γ be a C^1 simple closed curve bounding a bounded open set $\Omega \subset \mathbb{C}$. Assume also $f : U \to \mathbb{C}$ is complex differentiable with continuous derivative $f' \in C^0(U \to \mathbb{C})$ and assume $\overline{\Omega} \subset U$.

Given $z_0 \in \Omega$ and $\alpha : [a, b] \to \mathbb{C}$ is a C^1 parameterization of Γ, then

$$\begin{aligned} \int_\Gamma f &= \int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt \\ &= \int_a^b (u + iv)(\xi' + i\eta')\, dt \\ &= \int_a^b (u\xi' - v\eta')\, dt + i \int_a^b (u\eta' + v\xi')\, dt \end{aligned}$$

where $u = \operatorname{Re} f \circ \alpha(t)$, $v = \operatorname{Im} f \circ \alpha(t)$, $\xi' = \operatorname{Re} \alpha'(t)$, and $\eta' = \operatorname{Im} \alpha'(t)$.

Chapter 2

Analyticity

The main point of this chapter is to consider the power series expansion of a complex differentiable function $f : U \to \mathbb{C}$ where U is an open subset of $\mathbb{C}$. In order to write down a power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

one needs to have derivatives of all orders for the function f. Since we are only assuming complex differentiability, i.e., the existence of one complex derivative, the first order of business is to show the existence of all the other derivatives. Fortunately, we have a very powerful tool in the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}$$

where[1] $\overline{B_r(z_0)} \subset U$ and $z \in B_r(z_0)$.

2.1 Regularity

I will use the following result to get the existence of higher order derivatives:

Lemma 2.1. (Ahlfors' lemma) If $\phi \in C^0(\partial B_r(z_0) \to \mathbb{C})$ and $F_n : B_{r/2}(z_0) \to \mathbb{C}$ by

$$F_n(z) = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)^{n+1}},$$

[1] As usual $\partial B_r(z_0)$ is a circle with counterclockwise orientation (circumnavigated once).

then F_n is complex differentiable.

Proof: First consider the continuity of F_0. If $z_1 \in B_{r/2}(z_0)$ and $z \in B_{r_1}(z_1) \subset B_{r/2}(z_0)$, then

$$\begin{aligned} F_0(z) - F_0(z_1) &= \int_{\zeta \in \partial B_r(z_0)} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_1} \right) \phi(\zeta) \\ &= (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} |F_0(z) - F_0(z_1)| &= |z - z_1| \left| \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_1)} \right| \\ &\leq |z - z_1| \int_{\check{\mathbf{x}} \in \partial \check{B}_r(\mathbf{x}_0)} \frac{|\check{\phi}(\check{\mathbf{x}})|}{(r/2)^2} \qquad (2.1) \\ &= \frac{4}{r^2} \max_{\zeta \in \partial B_r(z_0)} |\phi(\zeta)| \; 2\pi r |z - z_1|. \end{aligned}$$

In (2.1) I've used some form of (1.7) which basically reduces to the case where we have a uniform bound on the integrand since

$$\left| \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_1)} \right| \leq \frac{4}{r^2} \max_{\zeta \in \partial B_r(z_0)} |\phi(\zeta)|.$$

The important point is that the number

$$M = \frac{4}{r^2} \max_{\zeta \in \partial B_r(z_0)} |\phi(\zeta)| \; 2\pi r = \frac{2\pi}{r} \max_{\zeta \in \partial B_r(z_0)} |\phi(\zeta)|$$

is a fixed nonnegative number independent of z. Consequently we have

$$|F_0(z) - F_0(z_1)| \leq M |z - z_1|$$

and this implies F_0 is continuous at z_1, and hence $F_0 \in C^0(B_{r/2}(z_0) \to \mathbb{C})$.

Next we consider differentiability for F_0. Again taking $z_1 \in B_{r/2}(z_0)$ the difference quotient

$$\frac{F_0(w) - F_0(z_1)}{w - z_1}$$

takes the form

$$\frac{F_0(w) - F_0(z_1)}{w - z_1} = \frac{1}{w - z_1} \int_{\zeta \in \partial B_r(z_0)} \left(\frac{1}{\zeta - w} - \frac{1}{\zeta - z_1} \right) \phi(\zeta)$$
$$= \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - w)(\zeta - z_1)}.$$

Taking $\tilde{\phi} : \partial B_r(z_0) \to \mathbb{C}$ by

$$\tilde{\phi}(\zeta) = \frac{\phi(\zeta)}{\zeta - z_1},$$

we see $\tilde{\phi} \in C^9(\partial B_r(z_0) \to \mathbb{C})$ and by the continuity argument above $\tilde{F}_1 : B_{r/2}(z_0) \to \mathbb{C}$ by

$$\tilde{F}_0(w) = \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{\zeta - w} = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - w)(\zeta - z_1)}$$

satisfies $\tilde{F}_0 \in C^0(B_{r/2}(z_0) \to \mathbb{C})$. In particular,

$$\lim_{w \to z_1} \frac{F_0(w) - F_0(z_1)}{w - z_1} = \lim_{w \to z_1} \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - w)(\zeta - z_1)} = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z_1)^2}$$

exists. We conclude F_0 is complex differentiable at z_1, and hence on $B_{r/2}(z_0)$ with

$$F_0'(z) = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)^2} = F_1(z).$$

In the case of a complex differentiable function (satisfying the Cauchy integral formula)

$$f(z) = \frac{1}{2\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z},$$

we now have not only a representation formula for the values of f on $B_r(z_0)$, but a representation formula for the values of the derivative f' on $B_{r/2}(z_0)$:

$$f'(z) = \frac{1}{2\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^2}. \tag{2.2}$$

It is particularly striking that the derivative f' does not appear under the integral sign on the right but only the values of f on $\partial B_r(z_0)$. Here of course we have simply taken $\phi(\zeta) = f(\zeta)/(2\pi i)$ in the proof above.

Exercise 2.1. Generalize the argument above to show that for any r_1 with $0 < r_1 < r$ one has $F_0 : B_{r_1}(z_0) \to \mathbb{C}$ complex differentiable with $F_0'(z) = F_1(z)$, and hence $F_0 : B_r(z_0) \to \mathbb{C}$ is complex differentiable with derivative given by the same formula.

I am going to carry out the details of a second step in the induction leading to the full proof of Ahlfors' lemma. As in the exercise above we fix r_1 with $0 < r_1 < r$. We will also use the result of Exercise 2.1 noting that $F_0' = F_1 \in C^0(B_r(z_0) \to \mathbb{C})$ since complex differentiability implies continuity.

Taking $z_1 \in B_{r_1}(z_0)$ and $z \in B_\delta(z_0)$ where $0 < \delta < (r - r_1)/2$ we find

$$\begin{aligned}
F_1(z) - F_1(z_1) &= \int_{\zeta \in \partial B_r(z_0)} \left(\frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - z_1)^2} \right) \phi(\zeta) \\
&= \int_{\zeta \in \partial B_r(z_0)} \left(\frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - z)(\zeta - z_1)} \right) \phi(\zeta) \\
&\qquad + \int_{\zeta \in \partial B_r(z_0)} \left(\frac{1}{(\zeta - z)(\zeta - z_1)} - \frac{1}{(\zeta - z_1)^2} \right) \phi(\zeta) \\
&= (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)^2(\zeta - z_1)} \\
&\qquad + (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_1)^2} \\
&= (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} \\
&\qquad + (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)} \\
&= (z - z_1) \left(\int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} + \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)} \right)
\end{aligned}$$

where $\tilde{\phi} \in C^0(\partial B_r(z_0) \to \mathbb{C})$ by

$$\tilde{\phi}(\zeta) = \frac{\phi(\zeta)}{\zeta - z_1}.$$

Note that in claiming $\tilde{\phi}$ is continuous we are using the fact that

$$|\zeta - z_1| \geq |\zeta - z_0| - |z_1 - z_0| = r - |z_1 - z_0| > r - r_1 > 0,$$

and thus $|\zeta - z_1|$ stays (uniformly) bounded away from zero.

For the continuity of F_1 at $z_1 \in B_{r_1}(z_0)$, we first focus on the integrand

$$\frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2}$$

and recall that $z \in B_\delta(z_1)$ here with $0 < \delta < (r - r_1)/2$. As just mentioned the numerator is continuous, and therefore the numerator $\tilde{\phi}$ is bounded in modulus by

$$\tilde{m} = \max_{\zeta \in \partial B_r(z_0)} |\tilde{\phi}(\zeta)| < \infty.$$

On the other hand, with respect to the denominator,

$$\begin{aligned} |\zeta - z| &\geq |\zeta - z_1| - |z - z_1| \\ &> |\zeta - z_0| - |z_1 - z_0| - \frac{r - r_1}{2} \\ &> r - r_1 - \frac{r - r_1}{2} \\ &= \frac{r - r_1}{2} \\ &> 0. \end{aligned}$$

Therefore,

$$\left| \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} \right| \leq \frac{4\tilde{m}}{(r - r_1)^2},$$

and

$$\left| \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} \right| \leq 2\pi r \frac{4\tilde{m}}{(r - r_1)^2} = \tilde{M}$$

with the bound here given by a fixed positive number independent of z. In particular

$$\left| (z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} \right| \leq \tilde{M} |z - z_1|.$$

The quantity

$$(z - z_1) \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)}$$

appearing in $F_1(z) - F_1(z_1)$ may be recognized as $\tilde{F}_0(z) - \tilde{F}_0(z_1)$ where

$$\tilde{F}_0(z) = \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{\zeta - z}.$$

By Exercise 2.1 applied to $\tilde{\phi}$, we know $\tilde{F}_0$ is continuous and

$$\lim_{z \to z_1} \tilde{F}_0(z) = \tilde{F}_0(z_1).$$

It follows that

$$|F_1(z) - F_1(z_1)| \leq \tilde{M}|z - z_1| + |\tilde{F}_0(z) - \tilde{F}_0(z_1)|,$$

so F_1 is continuous at z_1 and hence on $B_{r_1}(z_0)$ and hence on $B_r(z_0)$.

The calculation of $F_1(z) - F_1(z_1)$ above may also be used in consideration of the difference quotient

$$\frac{F_1(z) - F_1(z_1)}{z - z_1} = \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} + \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)}.$$

The first term here is

$$\tilde{F}_1(z) = \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2}.$$

Since $\tilde{\phi} \in C^0(\partial B_r(z_0) \to \mathbb{C})$ the continuity we have shown for F_1 applies also to $\tilde{F}_1$ and

$$\lim_{z \to z_1} \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^2} = \tilde{F}_1(z_1).$$

The second term

$$\int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)}$$

is precisely the difference quotient for $\tilde{F}_0$. Again, the argument above given for F_0 applies to $\tilde{F}_0$ so $\tilde{F}_0$ is differentiable with derivative

$$\lim_{z \to z_1} \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)(\zeta - z_1)} = \tilde{F}_1(z_1).$$

It now follows that F_1 is complex differentiable at z_1 with

$$F_1'(z_1) = 2\tilde{F}_1(z_1).$$

Note however that

$$\tilde{F}_1(z_1) = \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z_1)^2} = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z_1)^3} = F_2(z_1)$$

since

$$\tilde{\phi}(\zeta) = \frac{\phi(\zeta)}{\zeta - z_1}.$$

Thus, we have obtained along with the differentiability the formula

$$F_1'(z) = 2F_2(z).$$

Returning to the case of a complex differentiable function (satisfying the Cauchy integral formula)

$$f(z) = \frac{1}{2\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z},$$

we can use (2.2) in the lemma with

$$\phi(\zeta) = \frac{f(\zeta)}{2\pi i(\zeta - z)^2}$$

to get a formula for the complex derivative of f'. In fact, we know f is twice complex differentiable with

$$f''(z) = \frac{1}{\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^3}. \tag{2.3}$$

Notice formulas (2.2) and (2.3) are precisely what you would get by formally differentiating under the integral sign in Cauchy's integral formula.

Most importantly, this means a complex differentiable function has complex derivative which is again complex differentiable. Thus, the original function turns out to have derivatives of all orders. It is nice to have representation formulas for all these derivatives, so we continue with the proof of Lemma 2.1.

For the full induction we assume we have shown F_n is differentiable with

$$F_n'(z) = (n+1)\int_{\zeta\in\partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta-z)^{n+2}} = (n+1)F_{n+1}(z)$$

for $n = 0, 1, \ldots, k$ and all possible choices of $\phi \in C^0(\partial B_r(z_0) \to \mathbb{C})$.

As above taking r_1 with $0 < r_1 < r$ and points $z_1 \in B_{r_1}(z_0)$ and $z \in B_\delta(z_1)$ with $0 < \delta < (r - r_1)/2$, we consider

$$\begin{aligned}
F_{k+1}(z) - F_{k+1}(z_1) &= \int_{\zeta\in\partial B_r(z_0)} \left(\frac{1}{(\zeta-z)^{k+2}} - \frac{1}{(\zeta-z_1)^{k+2}}\right)\phi(\zeta) \\
&= \int_{\zeta\in\partial B_r(z_0)} \left(\frac{1}{(\zeta-z)^{k+2}} - \frac{1}{(\zeta-z)^{k+1}(\zeta-z_1)}\right)\phi(\zeta) \\
&\quad + \int_{\zeta\in\partial B_r(z_0)} \left(\frac{1}{(\zeta-z)^{k+1}(\zeta-z_1)} - \frac{1}{(\zeta-z_1)^{k+2}}\right)\phi(\zeta) \\
&= (z-z_1)\int_{\zeta\in\partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta-z)^{k+2}(\zeta-z_1)} \\
&\quad + \int_{\zeta\in\partial B_r(z_0)} \left(\frac{1}{(\zeta-z)^{k+1}} - \frac{1}{(\zeta-z_1)^{k+1}}\right)\tilde{\phi}(\zeta) \\
&= (z-z_1)\int_{\zeta\in\partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta-z)^{k+2}} + \tilde{F}_k(z) - \tilde{F}_k(z_1) \\
&= (z-z_1)\left(\int_{\zeta\in\partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta-z)^{k+2}} + \frac{\tilde{F}_k(z) - \tilde{F}_k(z_1)}{z-z_1}\right) \\
&= (z-z_1)\left(\tilde{F}_{k+1}(z) + \frac{\tilde{F}_k(z) - \tilde{F}_k(z_1)}{z-z_1}\right)
\end{aligned}$$

where $\tilde{\phi} \in C^0(\partial B_r(z_0) \to \mathbb{C})$ by

$$\tilde{\phi}(\zeta) = \frac{\phi(\zeta)}{\zeta - z_1}$$

and

$$\tilde{F}_k(z) = \int_{\zeta\in\partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta-z)^{k+1}}.$$

This gives

$$|F_{k+1}(z) - F_{k+1}(z_1)| \leq |z - z_1| \left| \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z)^{k+2}} \right| + |\tilde{F}_k(z) - \tilde{F}_k(z_1)|$$
$$\leq |z - z_1| N + |\tilde{F}_k(z) - \tilde{F}_k(z_1)|$$

where

$$N = 2\pi r \max \left\{ \frac{|\tilde{\phi}(\zeta)|}{|\zeta - z|^{k+2}} : \zeta \in \partial B_r(z_0),\ z \in B_\delta(z_1) \right\}.$$

In this way, we see F_{k+1} is continuous. This argument implies also $\tilde{F}_{k+1} \in C^0(B_r(z_0) \to \mathbb{C})$ with

$$\begin{aligned} \lim_{z \to z_1} \tilde{F}_{k+1}(z) &= \tilde{F}_{k+1}(z_1) \\ &= \int_{\zeta \in \partial B_r(z_0)} \frac{\tilde{\phi}(\zeta)}{(\zeta - z_1)^{k+2}} \\ &= \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z_1)^{k+3}} \\ &= F_{k+2}(z_1). \end{aligned}$$

Finally, for the difference quotent we have

$$\frac{F_{k+1}(z) - F_{k+1}(z_1)}{z - z_1} = \tilde{F}_{k+1}(z) + \frac{\tilde{F}_k(z) - \tilde{F}_k(z_1)}{z - z_1}.$$

The inductive hypothesis applies to the difference quotient

$$\frac{\tilde{F}_k(z) - \tilde{F}_k(z_1)}{z - z_1}$$

using $\tilde{\phi}$ in place of ϕ, so

$$\tilde{F}_k'(z_1) = \lim_{z \to z_1} \frac{\tilde{F}_k(z) - \tilde{F}_k(z_1)}{z - z_1} = (k+1)\tilde{F}_{k+1}(z_1) = (k+1)F_{k+2}(z_1).$$

We conclude

$$\lim_{z \to z_1} \frac{F_{k+1}(z) - F_{k+1}(z_1)}{z - z_1}$$

exists with

$$\lim_{z \to z_1} \frac{F_{k+1}(z) - F_{k+1}(z_1)}{z - z_1} = F_{k+2}(z_1) + (k+1)F_{k+2}(z_1) = (k+2)F_{k+2}(z_1).$$

This completes the induction and the proof of Ahlfors' lemma.

2.2 Simple closed curves

A curve $\Gamma \subset \mathbb{C}$ is most generally the image of a parameterization $\alpha \in C^0(I \to \mathbb{C})$ where I is an interval in $\mathbb{R}$. If the interval $I = [a, b]$ is closed, then the image set Γ will be closed and bouded. If such a parameterization can be found with component functions $\alpha_1 = \operatorname{Re}\alpha$ and $\alpha_2 = \operatorname{Im}\alpha$ satisfying $\alpha_j \in C^1(I)$ for $j = 1, 2$, then we say Γ is a C^1 curve, and we can also say such a parameterization is C^1. We will mostly be interested in C^1 curves.

For complex integration one also needs Γ to have an **orientation**, though two orientations are possible. If $\tilde{\Gamma}$, sometimes denoted $-\Gamma$, is the "same" curve with the opposite orientation, then

$$\int_{\tilde{\Gamma}} f = -\int_{\Gamma} f.$$

Recall the curve Γ (assumed parameterized by α which imposes on or is compatible with an orientation of Γ) is **closed** if $\alpha(b) = \alpha(b)$. The use of the term "closed" here is quite different from the uses of the term "closed" above in regard to the fact that the continuous image of a "closed" interval is "closed." In those cases a "closed" interval is one containing its endpoints in contrast to an open interval $I = (a, b) = \{x \in \mathbb{R} : a < x < b\}$ or a half open interval $(a, b]$ or $[b, a)$ containing one endpoint but not the other. That the image is "closed" means the compement $\mathbb{C}\backslash\Gamma$ is **open**. These uses for intervals and general sets are compatible, but what we are saying now is something different.

If a closed curve Γ has (a) parameterization $\alpha \in C^0([a, b] \to \mathbb{C})$ which is injective on $[a, b)$ and $(a, b]$, then Γ is said to be a **simple closed curve**. In order to be a C^1 simple closed curve, one also requires the existence of a C^1 parameterization for which $\alpha'(a) = \alpha'(b)$ where $\alpha' = \alpha_1' + i\alpha_2'$ is the real derivative of α.

A simple closed curve is also called a Jordan curve after the French mathematician Camille Jordan. Here is a difficult and important theorem which hopefully is easy to believe:

Theorem 4. (Jordan curve theorem in $\mathbb{C}$) Given a simple closed curve $\Gamma \subset \mathbb{C}$ the complement $\mathbb{C}\backslash\Gamma$ consists of precisely two open components Ω_{in} and Ω_{out} with Ω_{in} (metrically) bounded and Ω_{out} unbounded. Asice from the partitioning

$$\mathbb{C} = \Omega_{\text{in}} \cup \Gamma \cup \Omega_{\text{out}}$$

there also holds

$$\partial\Omega_{\text{in}} = \Gamma = \partial\Omega_{\text{out}}.$$

Ahlfors' lemma extends to C^1 simple closed curves as follows:

Lemma 2.2. If Γ is a C^1 simple closed curve, $\phi \in C^0(\Gamma \to \mathbb{C})$, then the function $F_n : \Omega_{\text{in}} \to \mathbb{C}$ by

$$F_n(z) = \int_{\zeta\in\Gamma} \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}$$

is complex differentiable with $F_n' = (n+1)F_{n+1}$.

Note: The set Ω_{in} in this version of Ahlfors' lemma can be replaced by Ω_{out} or simply $\mathbb{C}\backslash\Gamma$.

We can restate The Cauchy integral formula as follows:

2.3 Taylor coefficients