Math 6701, Exam 2: Name/Section: ____________________

1. (4.8.16) Here is a table of data for a function which is theoretically predicted to have the form $f(x) = ax^p$.

x	1.2	1.3	1.4	1.8	2.1	2.2
$f(x)$	2.6	3	3.3	4.8	6.1	6.5

We wish to find the best power p and coefficient a.

(a) (5 points) Compile a table of values of $\ln(ax^p)$ as a function of $\ln(x)$. You do not need to find decimal approximations. For example, when $\xi = \ln(1.2)$, then the corresponding value is $\ln(2.6) \approx \ln(a(1.2)^p)$.

ξ	$\ln(1.2)$					
$g(\xi)$	$\ln(2.6)$					

(b) (5 points) Formulate a linear algebra problem whose solution would be the vector $(\ln a, p)^T$ if there was a perfect fit. In other words, find a matrix A and a vector $\mathbf{b}$ such that

$$A \begin{pmatrix} \ln a \\ p \end{pmatrix} = \mathbf{b}$$

if it were the case that $2.6 = a(1.2)^p$, $3 = a(1.3)^p$, and so on.

(c) (5 points) Does the problem you formulated in part (b) have a solution? (Justify your answer. You may wish to use the following numerical apprximations (correct to four places):

$$\ln(3.3/2.6)/\ln(1.4/1.2) \approx 1.5466,$$
$$\ln(3/2.6)/\ln(1.3/1.2) \approx 1.7878.$$

(d) (10 points) Using the matrix A you defined in part (b), formulate a linear algebra problem which has a solution and gives the best fit values $\ln a$ and p. (You do not need to solve the problem on this exam, but describe the solution in terms of A and **b**.)

Solution:

(a) This is easy:

ξ	$\ln(1.2)$	$\ln(1.3)$	$\ln(1.4)$	$\ln(1.8)$	$\ln(2.1)$	$\ln(2.2)$
$g(\xi)$	$\ln(2.6)$	$\ln(3)$	$\ln(3.3)$	$\ln(4.8)$	$\ln(6.1)$	$\ln(6.5)$

(b) We would like to have $\ln(2.6) = \ln(a(1.2)^p) = \ln a + p\ln(1.2)$, and $\ln a + \xi p = g(\xi)$ in general. Therefore, a perfect fit would satisfy $\ln a + p\ln(1.2) = \ln(2.6)$, $\ln a + p\ln(1.3) = \ln(3)$, etc., that is,

$$\begin{pmatrix} 1 & \ln(1.2) \\ 1 & \ln(1.3) \\ 1 & \ln(1.4) \\ 1 & \ln(1.8) \\ 1 & \ln(2.1) \\ 1 & \ln(2.2) \end{pmatrix} \begin{pmatrix} \ln a \\ p \end{pmatrix} = \begin{pmatrix} \ln(2.6) \\ \ln(3) \\ \ln(3.3) \\ \ln(4.8) \\ \ln(6.1) \\ \ln(6.5) \end{pmatrix}.$$

(c) This problem has no solution. The first three rows of the coefficient matrix reduce as follows:

$$\begin{pmatrix} 1 & \ln(1.2) & \ln(2.6) \\ 0 & \ln(1.3/1.2) & \ln(3/2.6) \\ 0 & \ln(1.4/1.2) & \ln(3.3/2.6) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \ln(1.2) & \ln(2.6) \\ 0 & 1 & \ln(3.2/2.6)/\ln(1.3/1.2) \\ 0 & 1 & \ln(3.3/2.6)/\ln(1.4/1.2) \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \ln(1.2) & \ln(2.6) \\ 0 & 1 & \ln(3.2/2.6)/\ln(1.3/1.2) \\ 0 & 0 & \ln(3.3/2.6)/\ln(1.4/1.2) - \ln(3.2/2.6)/\ln(1.3/1.2) \end{pmatrix}$$

The last equation is inconsistent by the stated approximations.

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(d) Given A and $\mathbf{b}$ as defined in part (b), the least squares approximate solution of the problem stated in (b) is the solution of

$$A^T A \begin{pmatrix} u \\ p \end{pmatrix} = A^T \mathbf{b},$$

or

$$\begin{pmatrix} u \\ p \end{pmatrix} = (A^T A)^{-1} A^T \mathbf{b},$$

and we take coefficient $a = e^u$ and power p.

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2. (25 points) (8.2.29) The water in a lake has reached a 90% contamination level. Pure water is pumped into the lake at 1000 liters per minute, and the lake overflows into a stream at the same rate. Let $p = p(t)$ denote the contaminant in the lake as a function of time. Assume the water entering the stream contains $p(t)/10^{10}$ contaminant per liter, and determine how long it will take for the stream to run with 50% contaminant.

Solution: The contaminant satisfies

$$\frac{d}{dt}p = -10^3 \frac{p}{10^{10}} \qquad \text{and} \qquad p(0) = (9)10^9.$$

This tells us $p(t) = (9)10^9)e^{-t/10^7}$.

We want to know when this quantity is $10^{10}/2$. That is, when

$$t = -10^7 \ln(5/9).$$

This is in minutes. So that would be a little over 11 years.

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3. (25 points) (8.6.33) An undamped oscillator $L[y] = y'' + y$ is driven at frequency ω by the forcing term $f(t) = \cos(\omega t)$. We say the forcing is *at the resonant frequency* if the resulting motion is unbounded. What is the resonant freqency? (Justify your answer.)

Solution: The general solution of the associated homogeneous ODE is

$$y_h(x) = a\cos t + b\sin t.$$

This function is clearly bounded. If $\omega \neq 1$, we can find a particular solution of the form $y_p = A\cos\omega t + B\sin\omega t$ which will also be bounded. Thus, the only possible resonant frequency is $\omega = 1$. In that case,

$$y = a\cos t + b\sin t + t(A\cos t + B\sin t)$$

which will certainly be unbounded regardless of the choice of A and B (since they can't both be zero).

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4. (25 points) (8.11.7) A damped oscillator $L[y] = y'' + y' + y$ is set in motion with

$$y(0) = \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} \qquad \text{and} \qquad y'(0) = -\frac{1}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}},$$

and experiences a unit impulse at time $t = 5\pi/\sqrt{3}$. Describe the resulting motion $y(t)$.

Solution: Let $\mathcal{L}[y] = Y$ be the Laplace transform of y. Then

$$s^2 Y - s\frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} + \frac{1}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} + sY - \frac{2}{\sqrt{3}} e^{\frac{5\pi}{\sqrt{3}}} + Y = e^{\frac{5\pi}{2\sqrt{3}}s}$$

or

$$(s^2 + s + 1)Y = \frac{1}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}}(2s+1) + e^{\frac{5\pi}{\sqrt{3}}s}.$$

That is,

$$\begin{aligned}
Y &= \frac{1}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} \frac{2s+1}{(s+1/2)^2 + 3/4} + \frac{e^{\frac{5\pi}{\sqrt{3}}s}}{(s+1/2)^2+3/4} \\
&= \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} \mathcal{L}\left[e^{-t/2}\cos\frac{\sqrt{3}}{2}t\right] + \frac{2}{\sqrt{3}} e^{\frac{5\pi}{\sqrt{3}}s} \mathcal{L}\left[e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right] \\
&= \frac{2}{\sqrt{3}}\left\{ e^{\frac{5\pi}{2\sqrt{3}}} \mathcal{L}\left[e^{-t/2}\cos\frac{\sqrt{3}}{2}t\right] + \mathcal{L}\left[u\left(t - \frac{5\pi}{\sqrt{3}}\right) e^{-\left(t-\frac{5\pi}{\sqrt{3}}\right)/2} \sin\left(\frac{\sqrt{3}}{2}t - \frac{5\pi}{2}\right)\right]\right\} \\
&= \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}}\left\{ \mathcal{L}\left[e^{-t/2}\cos\frac{\sqrt{3}}{2}t\right] + \mathcal{L}\left[u\left(t - \frac{5\pi}{\sqrt{3}}\right) e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{2}\right)\right]\right\} \\
&= \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} \mathcal{L}\left[e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + u\left(t - \frac{5\pi}{\sqrt{3}}\right)\left(-\cos\frac{\sqrt{3}}{2}t\right)\right)\right] \\
&= \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} \mathcal{L}\left[e^{-t/2}\cos\frac{\sqrt{3}}{2}t\left(1 - u\left(t - \frac{5\pi}{\sqrt{3}}\right)\right)\right].
\end{aligned}$$

Thus,

$$y(t) = \frac{2}{\sqrt{3}} e^{\frac{5\pi}{2\sqrt{3}}} e^{-t/2} \cos\frac{\sqrt{3}}{2}t\left(1 - u\left(t - \frac{5\pi}{\sqrt{3}}\right)\right)$$

is a decaying exponential until the impulse, at which time all motion ceases, and the system remains in equilibrium $y = 0$ after $y = 5\pi/\sqrt{3}$.