Exam 2
Due Friday October 16, 2020

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Problem 1 *Recall the complex numbers are given by*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

(a) *Show that* $\mathbb{C}$ *is a vector space over* $\mathbb{R}$*. What is the dimension?*

Usually, when we consider $\mathbb{C}$ *as a vector space we assume it is considered as a vector space of dimension one over* $\mathbb{C}$*. Let us denote the vector space* $\mathbb{C}$ *as a vector space over* $\mathbb{R}$ *by* $\mathbb{C}_{\mathbb{R}}$*.*

(b) *Let* $\mathcal{L}(\mathbb{C})$ *denote the collection of all linear functions* $L : \mathbb{C} \to \mathbb{C}$*. You should have characterized this collection in Problem 1 of Assignment 4. Show that* $\mathcal{L}(\mathbb{C})$ *is a vector space over* $\mathbb{C}$*. What is the dimension?*

(c) *Let* $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ *denote the collection of all linear functions* $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$*. Show that* $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ *is a vector space over* $\mathbb{R}$*. What is the dimension of* $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$*?*

(c) *Can you compare* $\mathcal{L}(\mathbb{C})$ *and* $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$*? Hint: Can one be realized as a* **subset** *of the other? What happens to the algebraic properties?*

Problem 2 *Consider the two dimensional vector space of polynomials of degree less than or equal to one:*

$$P_1[x] = \{ax + b : a, b \in \mathbb{R}\}.$$

Let $L : P_1[x] \to P_1[x]$ *by*

$$L[ax + b] = (a + b)x + b.$$

(a) *Express* L *in terms of differentiation. Hint: Note that if* $p(x) = ax + b$*, then* $a = p'(x)$*. (So what is* b*?)*

(b) *Show that* L *is linear.*

(c) *Express* L *in terms of matrix multiplication with respect to the basis* $\mathcal{B}_1 = \{x, 1\}$ *(for both the domain and the co-domain).*

(d) *Show that* $\mathcal{B}_2 = \{x + 1, 3\}$ *is a basis for* $P_1[x]$*.*

(e) *Consider the linear transformation* $T : P_1[x] \to P_1[x]$ *given by matrix multiplication with the matrix*

$$\begin{pmatrix} 2 & 3 \\ -1/3 & 0 \end{pmatrix}$$

with respect to the basis $\mathcal{B}_2$ *(for both the domain and the co-domain). This means*

$$T[a(x + 1) + 3b] = (2a + 3b)(x + 1) - (a/3)(3)$$

because

$$\begin{pmatrix} 2 & 3 \\ -1/3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 3b \\ -a/3 \end{pmatrix}.$$

Calculate $T[ax + b]$*. What interesting thing does this tell you?*

Problem 3 *(Boas 3.11.35) Let* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear function. Show the following:*

> If there exists a basis $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$ such that the matrix of L with respect to $\mathcal{B}$ (for the domain and co-domain) is a symmetric matrix, then L has a real eigenvector.

Two more related questions:

(a) *What additional conditions are required to show there exists a basis for* $\mathbb{R}^2$ *consisting of eigenvectors of* L*?*

(b) *Is it possible that the matrix of* L *with respect to the standard unit basis* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is* **not** *symmetric?*

Problem 4 *(Boas 3.2.14) Consider the system of linear equations:*

$$\begin{cases} 2x + 3y - z = -2 \\ x + 2y - z = 4 \\ 4x + 7y - 3z = 11. \end{cases}$$

(a) *Write down the equivalent matrix equation and identify the coefficient matrix, the unknown vector, and the inhomogeneity.*

(b) *Find a basis for the column space consisting of columns of the coefficient matrix.*

(c) *Find a basis for the solution space of the associated homogeneous equation.*

(d) *What is the solution set of the (original) system?*

Problem 5 *(Boas 3.11.13) Consider* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ 2x - y \end{pmatrix}.$$

(a) *Find the eigenvalues of* L *by considering the equation*

$$\det\begin{pmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{pmatrix} = 0.$$

(b) *Consider* $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $\Lambda(\xi, \eta)^T = T^{-1} \circ L \circ T(\xi, \eta)^T$ *where* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \xi + 2\eta \\ -2\xi + \eta \end{pmatrix}.$$

(The linear transformations L *and* Λ *are said to be* **conjugate** *or* **similar***).*

(i) *Draw* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

(ii) *Use computational software to draw* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

(iii) *Use computational software to draw* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

(iv) *Find an equation of the form*

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

satisfied by the points in $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$. *(Determine the values of* a *and* b.

(v) *Find an equation of the form*

$$Ax^2 + Bxy + Cy^2 = 180$$

satisfied by the points in $\{L(x, y)^T : x^2 + y^2 = 1\}$.

(c) *Define a function* $e^L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$e^L \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{j=0}^{\infty} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix}$$

where L^j *is iteration, meaning apply L over and over again* j *times, as usual.*

(i) *Find the values of*

$$L^2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad L^3 \begin{pmatrix} x \\ y \end{pmatrix}.$$

If you do this directly, it should convince you that it is not exactly straightforward to find the value of e^L*. You can complete the remaining parts of this problem to find that value.*

(ii) *Find the value of*

$$\Lambda^j \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

(iii) *Notice that* $L^j(x, y)^T = T \circ \Lambda^j \circ T^{-1}(x, y)^T$.

(iv) *Notice that the partial sum*

$$\sum_{j=1}^{k} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix} = T \circ \left(\sum_{j=1}^{k} \frac{1}{j!} \Lambda^j \right) \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(v) *Find the value of*

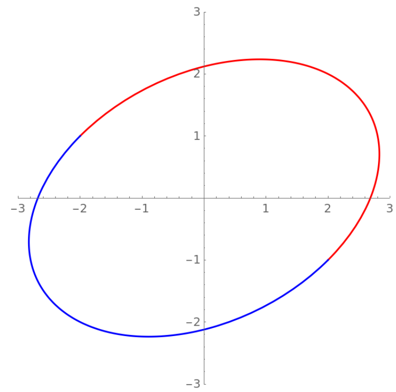
$$e^L \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution:

(a) $(2 - \lambda)(-1 - \lambda) - 4 = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3)$, so the eigenvalues are $\lambda = -2$ and $\lambda = 3$.

(b) Matrices in Mathematica:

```
Mu = {{2, 2}, {2, -1}}
Tu = {{1, 2}, {-2, 1}}/Sqrt[5]
Tuinv=Inverse[Tu]
```

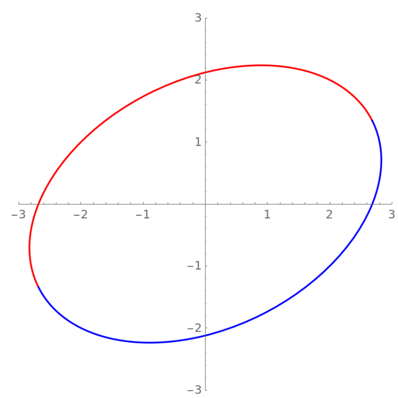


(i)

Figure 1: The image of the unit circle under L using the matrix M (capital mu).

Mathematica command:

```
Show[
ParametricPlot[Mu.{Cos[t], Sin[t]}, {t, -Pi/2, Pi/2},
PlotStyle -> Red, PlotRange -> {{-3, 3}, {-3, 3}}],
ParametricPlot[Mu.{Cos[t], Sin[t]}, {t, Pi/2, 3 Pi/2},
PlotStyle -> Blue]
]
```



(ii)

Figure 2: The image of the unit circle under $L \circ T$.

Mathematica command:

```
Show[
```

```
ParametricPlot[(Mu.Tu).{Cos[t], Sin[t]}, {t, -Pi/2, Pi/2},
PlotStyle -> Red, PlotRange -> {{-3, 3}, {-3, 3}}],
ParametricPlot[(Mu.Tu).{Cos[t], Sin[t]}, {t, Pi/2, 3 Pi/2},
PlotStyle -> Blue]
]
```

(iii)

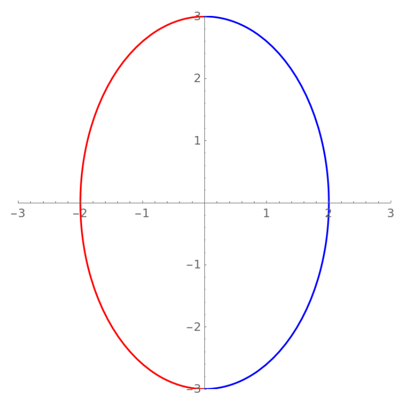


Figure 3: The image of the unit circle under $T^{-1} \circ L \circ T$.

Mathematica command:

```
Show[
ParametricPlot[(Tuinv.Mu.Tu).{Cos[t], Sin[t]}, {t, -Pi/2, Pi/2},
PlotStyle -> Red, PlotRange -> {{-3, 3}, {-3, 3}}],
ParametricPlot[(Tuinv.Mu.Tu).{Cos[t], Sin[t]}, {t, Pi/2, 3 Pi/2},
PlotStyle -> Blue]
]
```

(iv) Setting

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1} \circ L \circ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

we see

$$\frac{\xi^2}{4} + \frac{\eta^2}{9} = x^2 + y^2 = 1$$

(as long as $x^2 + y^2 = 1$). Thus, $a = 2$ and $b = 3$.

(v) Note that

$$\left\{ L \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1 \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \left| L^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right| = 1 \right\}.$$

Since

$$L^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

we find

$$\left| L^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right|^2 = \frac{1}{36}[(x + 2y)^2 + (2x - 2y)^2]$$

and the equation is

$$5x^2 - 4xy + 8y^2 = 36.$$

I'm not sure what the point of the 108 is, but this can also be written as

$$15x^2 - 12xy + 24y^2 = 108.$$

(c) Define a function $e^L : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$e^L \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{j=0}^{\infty} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix}.$$

(i)

$$L^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$L^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 & 14 \\ 14 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The point is that no obvious pattern arises in computing these powers/iterates.

(ii) In contrast

$$\Lambda^j \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (-2)^j & 0 \\ 0 & 3^j \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

(iii)

$$
\begin{aligned}
L^j \begin{pmatrix} x \\ y \end{pmatrix} &= (T \circ \Lambda \circ T^{-1})^j \begin{pmatrix} x \\ y \end{pmatrix} \\
&= (T \circ \Lambda \circ T^{-1}) \circ (T \circ \Lambda \circ T^{-1}) \circ \cdots \circ (T \circ \Lambda \circ T^{-1}) \begin{pmatrix} x \\ y \end{pmatrix} \\
&= T \circ \Lambda \circ (T^{-1} \circ T) \circ \Lambda \circ (T^{-1} \circ T) \circ \Lambda \circ \cdots \circ (T^{-1} \circ T) \circ \Lambda \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= T \circ \Lambda^j \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\sum_{j=1}^{k} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix} &= \sum_{j=1}^{k} \frac{1}{j!} T \circ \Lambda^j \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= T \circ \left(\sum_{j=1}^{k} \frac{1}{j!} \Lambda^j \right) \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

(v)

$$
\begin{aligned}
e^L \begin{pmatrix} x \\ y \end{pmatrix} &= \sum_{j=1}^{\infty} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix} \\
&= T \left(\sum_{j=1}^{\infty} \frac{1}{j!} \begin{pmatrix} (-2)^j & 0 \\ 0 & 3^j \end{pmatrix} T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\
&= T \left(\begin{pmatrix} \sum_{j=1}^{\infty} \frac{(-2)^j}{j!} & 0 \\ 0 & \sum_{j=1}^{\infty} \frac{3^j}{j!} \end{pmatrix} T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\
&= T \left(\begin{pmatrix} e^{-2} & 0 \\ 0 & e^3 \end{pmatrix} \left(\frac{1}{\sqrt{5}} \begin{pmatrix} x - 2y \\ 2x + y \end{pmatrix} \right) \right) \\
&= \frac{1}{\sqrt{5}} T \begin{pmatrix} (x-2y)e^{-2} \\ (2x+y)e^3 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} (x-2y)e^{-2} + 2(2x+y)e^3 \\ -2(x-2y)e^{-2} + (2x+y)e^3 \end{pmatrix}.
\end{aligned}
$$

Thus, $e^L : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$e^L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-2} + 4e^3 & -2e^{-2} + 2e^3 \\ -2e^{-2} + 2e^3 & 4e^{-2} + e^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, the exponential of a diagonalizable transformation is a little complicated, but it can be computed. It is not too bad. Your question should be: How do you find the change of basis $T : \mathbb{R}^2 \to \mathbb{R}^2$?