# Exam 2
# Due Friday October 16, 2020

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October 14, 2020

**Problem 1** *Recall the complex numbers are given by*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

**(a)** *Show that $\mathbb{C}$ is a vector space over $\mathbb{R}$. What is the dimension?*

*Usually, when we consider $\mathbb{C}$ as a vector space we assume it is considered as a vector space of dimension one over $\mathbb{C}$. Let us denote the vector space $\mathbb{C}$ as a vector space over $\mathbb{R}$ by $\mathbb{C}_{\mathbb{R}}$.*

**(b)** *Let $\mathcal{L}(\mathbb{C})$ denote the collection of all linear functions $L : \mathbb{C} \to \mathbb{C}$. You should have characterized this collection in Problem 1 of Assignment 4. Show that $\mathcal{L}(\mathbb{C})$ is a vector space over $\mathbb{C}$. What is the dimension?*

**(c)** *Let $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ denote the collection of all linear functions $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$. Show that $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ is a vector space over $\mathbb{R}$. What is the dimension of $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$?*

**(c)** *Can you compare $\mathcal{L}(\mathbb{C})$ and $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$? Hint: Can one be realized as a* **subset** *of the other? What happens to the algebraic properties?*

**Problem 2** *Consider the two dimensional vector space of polynomials of degree less than or equal to one:*

$$P_1[x] = \{ax + b : a, b \in \mathbb{R}\}.$$

*Let* $L : P_1[x] \to P_1[x]$ *by*

$$L[ax + b] = (a + b)x + b.$$

**(a)** *Express* $L$ *in terms of differentiation. Hint: Note that if* $p(x) = ax + b$*, then* $a = p'(x)$*. (So what is* $b$*?)*

**(b)** *Show that* $L$ *is linear.*

**(c)** *Express* $L$ *in terms of matrix multiplication with respect to the basis* $\mathcal{B}_1 = \{x, 1\}$ *(for both the domain and the co-domain).*

**(d)** *Show that* $\mathcal{B}_2 = \{x + 1, 3\}$ *is a basis for* $P_1[x]$.

**(e)** *Consider the linear transformation* $T : P_1[x] \to P_1[x]$ *given by matrix multiplication with the matrix*

$$\begin{pmatrix} 2 & 3 \\ -1/3 & 0 \end{pmatrix}$$

*with respect to the basis* $\mathcal{B}_2$ *(for both the domain and the co-domain). This means*

$$T[a(x + 1) + 3b] = (2a + 3b)(x + 1) - (a/3)(3)$$

*because*

$$\begin{pmatrix} 2 & 3 \\ -1/3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 3b \\ -a/3 \end{pmatrix}.$$

*Calculate* $T[ax + b]$*. What interesting thing does this tell you?*

**Problem 3** *(Boas 3.11.35) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function. Show the following:*

> If there exists a basis $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$ such that the matrix of $L$ with respect to $\mathcal{B}$ (for the domain and co-domain) is a symmetric matrix, then $L$ has a real eigenvector.

*Two more related questions:*

**(a)** *What additional conditions are required to show there exists a basis for $\mathbb{R}^2$ consisting of eigenvectors of $L$?*

**(b)** *Is it possible that the matrix of $L$ with respect to the standard unit basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is* **not** *symmetric?*

**Problem 4** *(Boas 3.2.14) Consider the system of linear equations:*

$$\begin{cases} 2x + 3y - z = -2 \\ x + 2y - z = 4 \\ 4x + 7y - 3z = 11. \end{cases}$$

**(a)** *Write down the equivalent matrix equation and identify the coefficient matrix, the unknown vector, and the inhomogeneity.*

**(b)** *Find a basis for the column space consisting of columns of the coefficient matrix.*

**(c)** *Find a basis for the solution space of the associated homogeneous equation.*

**(d)** *What is the solution set of the (original) system?*

**Problem 5** *(Boas 3.11.13) Consider* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ 2x - y \end{pmatrix}.$$

**(a)** *Find the eigenvalues of* $L$ *by considering the equation*

$$\det \begin{pmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{pmatrix} = 0.$$

**(b)** *Consider* $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $\Lambda(\xi, \eta)^T = T^{-1} \circ L \circ T(\xi, \eta)^T$ *where* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \xi + 2\eta \\ -2\xi + \eta \end{pmatrix}.$$

*(The linear transformations* $L$ *and* $\Lambda$ *are said to be* **conjugate** *or* **similar***).*

**(i)** *Draw* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

**(ii)** *Use computational software to draw* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

**(iii)** *Use computational software to draw* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$ *with* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi > 0\}$ *red and* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1, \xi < 0\}$ *blue.*

**(iv)** *Find an equation of the form*

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

*satisfied by the points in* $\{T^{-1} \circ L \circ T(\xi, \eta)^T : \xi^2 + \eta^2 = 1\}$*. (Determine the values of* $a$ *and* $b$*.*

**(v)** *Find an equation of the form*

$$Ax^2 + Bxy + Cy^2 = 180$$

*satisfied by the points in* $\{L(x, y)^t : x^2 + y^2 = 1\}$*.*

**(c)** *Define a function* $e^L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$e^L \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{j=0}^{\infty} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix}$$

*where* $L^j$ *is iteration, meaning apply* $L$ *over and over again* $j$ *times, as usual.*

**(i)** *Find the values of*

$$L^2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad L^3 \begin{pmatrix} x \\ y \end{pmatrix}.$$

*If you do this directly, it should convince you that it is not exactly straightforward to find the value of* $e^L$*. You can complete the remaining parts of this problem to find that value.*

**(ii)** *Find the value of*

$$\Lambda^j \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \textit{for } k = 0, 1, 2, 3, \ldots.$$

**(iii)** *Notice that* $L^j(x,y)^T = T \circ \Lambda^j \circ T^{-1}(x,y)^T$.

**(iv)** *Notice that the partial sum*

$$\sum_{j=1}^{k} \frac{1}{j!} L^j \begin{pmatrix} x \\ y \end{pmatrix} = T \circ \left( \sum_{j=1}^{k} \frac{1}{j!} \Lambda^j \right) \circ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

**(v)** *Find the value of*

$$e^L \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Problem 6** *Consider* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 3y \end{pmatrix}.$$

**(a)** *Use computational software to draw* $\{L(x, y)^T : x^2 + y^2 = 1\}$ *with* $\{L(x, y)^T : x^2 + y^2 = 1, x > 0\}$ *red and* $\{L(x, y)^T : x^2 + y^2 = 1, x < 0\}$ *blue.*

**(b)** *Draw* $L(1, 0)^T$ *and* $L(1/\sqrt{2}, 1/\sqrt{2})^T$ *on your figure.*

**(c)** *What kind of set/curve do you think* $\{L(x, y)^T : x^2 + y^2 = 1\}$ *is? Can you prove your assertion?*

**(d)** *Find a diagonal linear transformation* $\Lambda$ *which is similar to* $L$*. (This should convince you that similarity does not preserve the geometry of a linear function. In other words, because you understand the geometry of a linear transformation* $\Lambda$ *which is similar to* $L$*, does not mean you understand the geometry of the mapping* $L$*.)*

**Problem 7** *Consider* $L : \mathbb{C}^2 \to \mathbb{C}^2$ *given by*

$$L \begin{pmatrix} z \\ w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z - w\sqrt{3} \\ z\sqrt{3} + w \end{pmatrix}$$

*A complex eigenvector for* $L$ *is a* **nonzero** *vector* $\mathbf{z} \in \mathbb{C}^2$ *for which there is a complex number* $\lambda + i\mu$ *with*

$$L(\mathbf{z}) = (\lambda + i\mu)\mathbf{z}.$$

**(a)** *Find an eigenvector* $\mathbf{z} = \mathrm{Re}(\mathbf{z}) + i\,\mathrm{Im}(\mathbf{z})$ *for* $L$*.*

**(b)** *Show that* $\bar{\mathbf{z}} = \mathrm{Re}(\mathbf{z}) - i\,\mathrm{Im}(\mathbf{z})$ *is an eigenvector for* $L$*. What is the eigenvalue associated with* $\bar{\mathbf{z}}$*?*

**(c)** *Show that* $\{\mathbf{z}, \bar{\mathbf{z}}\}$ *is a basis for* $\mathbb{C}^2$*.*

**Problem 8** *Draw mapping pictures to illustrate the linear functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $L\mathbf{x} = A\mathbf{x}$ *for the following matrices* $A$*:*

**(a)**

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1 < -1 < \lambda_2 < 0.$$

**(b)**

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{with} \qquad \lambda < 0.$$

**(c)**

$$A = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \qquad \text{with} \qquad 0 < \lambda < \mu.$$

*Hint: In case (c) write* $A$ *as*

$$\alpha \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

**Problem 9** *The "forms" given in parts (a), (b), and (c) of Problem 8 are three canonical forms for matrices, meaning that given any $2 \times 2$ matrix $A_0$, there is an invertible (change of basis) matrix $Q$ for which*

$$QA_0Q^{-1} = A$$

*with $A$ in one of the "forms" given in Problem 8 (not counting the conditions on the real values of $\lambda_1$, $\lambda_2$, $\lambda$, or $\mu$). The matrices $A_0$ and $A$ are said to be conjugate or similar.*

**(a)** *Find the change of basis matrix $Q$ for which the matrix of the linear function $L$ from Problem 7 (with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$) is conjugate to a canonical form matrix.*

**(b)** *Recall that the linear function from Problem 7 had domain and co-domain $\mathbb{C}^2$. The canonical forms from Problem 8 apply to functions with domain and co-domain $\mathbb{R}^2$. Is $\{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}$ a subspace of $\mathbb{C}^2$?*

**(c)** *Find a subset $A \subset \mathbb{R}^2$ which is closed under addition but not under scaling and a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ for which the restriction to $A$ gives a well-defined function $L_0 : A \to A$. In fact, find such an example for which $A$ is a vector space over a subfield of $\mathbb{R}$ and $L_0$ is linear with respect to this vector space. Hint: $\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

**Problem 10** *(Boas Chapter 3 Sections 8 and 14) Given a vector space $V$ over $\mathbb{R}$, an* **inner product** *is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying the following properties:*

**(i)** *(symmetric) $\langle v, w \rangle = \langle w, v \rangle$ for every $v, w \in V$.*

**(ii)** *(bilinear[1]) $\langle av + bw, z \rangle = a\langle v, z \rangle + b\langle w, z \rangle$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.*

**(iii)** *(positive definite) $\langle v, v \rangle \geq 0$ for every $v \in V$ with equality if and only if $v = \mathbf{0}$.*

*Whenever there is an inner product on a real vector space $V$, then the* **length** *or* **norm** *of any vector $v \in V$ is defined by*

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

*Whenever there is an inner product on a real vector space $V$, two vectors $v$ and $w$, are said to be* **orthogonal** *or* **perpendicular** *if $\langle v, w \rangle = 0$.*

**(a)** *Check that the dot product on $\mathbb{R}^n$ given by*

$$\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{n} v_j w_j$$

*is an inner product.*

**(b)** *Given any real vector space $V$ with an inner product, show*

$$\left\| \frac{v}{\|v\|} \right\| = 1 \qquad \text{for any } v \in V \setminus \{\mathbf{0}\},$$

*that is, $v/\|v\|$ is a* **unit vector**.

**(c)** *Let $C^0[0, 2\pi]$ denote the real valued continuous functions on the interval $[0, 2\pi]$ as usual. Recall that this is a vector space over $\mathbb{R}$. Show that*

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$$

*defines an inner product on $C^0[0, 1]$. This is called the $L^2$ inner product.*

[1] Technically condition (ii) is only linearity in the first argument, but in view of the symmetry condition, an inner product is linear in both arguments, or **bilinear**.

**(d)** *Consider the order one* **Fourier polynomials** *given by*

$$F_1[x] = \mathrm{span}\{\cos x, \sin x, 1\} \subset C^0[0, 2\pi].$$

*Find an orthonormal basis for* $F_1[x]$*. Note: These functions* $a\cos x + b\sin x + c$ *are not polynomials in the usual sense, but still they are called Fourier polynomials. Second order Fourier polynomials would look like* $a_2\cos(2x) + b_2\sin(2x) + a_1\cos x + b_1\sin x + c$*.*

**(e)** *Find the projection of the function* $f \in C^0[0, 2\pi]$ *given by* $f(x) = x$ *onto the subspace* $F_1[x]$*.*