Exam 2
Due Friday October 16, 2020

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Problem 1 *Recall the complex numbers are given by*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

(a) *Show that $\mathbb{C}$ is a vector space over $\mathbb{R}$. What is the dimension?*

Usually, when we consider $\mathbb{C}$ as a vector space we assume it is considered as a vector space of dimension one over $\mathbb{C}$. Let us denote the vector space $\mathbb{C}$ as a vector space over $\mathbb{R}$ by $\mathbb{C}_{\mathbb{R}}$.

(b) *Let $\mathcal{L}(\mathbb{C})$ denote the collection of all linear functions $L : \mathbb{C} \to \mathbb{C}$. You should have characterized this collection in Problem 1 of Assignment 4. Show that $\mathcal{L}(\mathbb{C})$ is a vector space over $\mathbb{C}$. What is the dimension?*

(c) *Let $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ denote the collection of all linear functions $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$. Show that $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ is a vector space over $\mathbb{R}$. What is the dimension of $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$?*

(d) *Can you compare $\mathcal{L}(\mathbb{C})$ and $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$? Hint: Can one be realized as a* **subset** *of the other? What happens to the algebraic properties?*

Solution:

(a) We know how to add and scale complex numbers, as $\mathbb{C}$ is a field. If we specialize the scaling to scaling by reals, then clearly the algebraic properties of the field (associative, commutative, distributive, identity, inverses, etc.) still hold. Therefore, $\mathbb{C}$ is a vector field over $\mathbb{R}$. A basis is given by

$$\mathcal{B} = \{1, i\}$$

as every complex number $x + iy$ (with $x, y \in \mathbb{R}$ can be written as a linear combination of these two complex numbers:

$$x + iy = x(1) + y(i)$$

with real coefficients. This means $\mathcal{B}$ is a spanning set. On the other hand, if the linear combination $x(1) + y(i) = 0$, then we know $x = y = 0$. That is, the set $\mathcal{B}$ is a linearly independent set. This shows the dimension of $\mathbb{C}$ over $\mathbb{R}$ is 2.

(b) Now we need to know how to add and scale linear functions $L : \mathbb{C} \to \mathbb{C}$. If L_1 and L_2 are linear functions, then we define the sum (as usual) by $L_1 + L_2 : \mathbb{C} \to \mathbb{C}$ by

$$(L_1 + L_2)(z) = L_1(z) + L_2(z).$$

Is this function linear? Yes it is:

$$\begin{aligned}(L_1 + L_2)(az + bw) &= L_1(az + bw) + L_2(az + bw) \\ &= aL_1z + bL_1w + aL_2z + bL_2w \\ &= a(L_1 + L_2)z + b(L_1 + L_2)w.\end{aligned}$$

We can also scale a linear function by any $a \in \mathbb{C}$:

$$(aL)(z) = aL(z)$$

and the result is also linear. Thus, we have addition and scaling. In particular, $(-1)L$ is an additive inverse for L with $L+(-L)$ giving the zero linear function $L_0 : \mathbb{C} \to \mathbb{C}$ by $L_0(z) \equiv 0$ which is, of course, a linear function and the additive inverse in our vector space $\mathcal{L}(\mathbb{C})$. The required algebraic properties are easy to check:

$$(L_1 + L_2) + L_3 = L_1 + (L_2 + L_3) \qquad \text{and} \qquad (ab)L = a(bL) \qquad \text{(associative)}$$

$$L_1 + L_2 = L_2 + L_1 \qquad \text{(commutative)}$$

$$(a + b)L = aL + bL \qquad \text{(distributive)}$$

These all follow directly from the corresponding properties of $\mathbb{C}$ as a field. We could write out the properties in more detail, but the main point is that linear functions can be considered as vectors.

Now, the interesting question: What is a basis for this vector space over $\mathbb{C}$? Actually, a set containing any nonzero element of $\mathcal{L}(\mathbb{C})$ will do, but there is one obvious choice for a basis element which is the identity transformation

$$\mathrm{id} : \mathbb{C} \to \mathbb{C} \qquad \text{by} \qquad \mathrm{id}(z) = z.$$

As observed in the wonderful Problem 1 of Assignment 4, given any linear function $L : \mathbb{C} \to \mathbb{C}$, we have $L(z) = zL(1)$. This means $L(z) = L(1)\,\mathrm{id}(z)$ or

$$L = L(1)\,\mathrm{id}\,.$$

Thus, $\{\mathrm{id}\}$ is a basis for $\mathcal{L}(\mathbb{C})$, and this vector space is one dimensional over $\mathbb{C}$.

(c) Every function $\mathcal{L}(\mathbb{C})$ is also in $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$: If a function $L : \mathbb{C} \to \mathbb{C}$ satisfies

$$L(az + bw) = aLz + bLw$$

for every $a, b, z, w \in \mathbb{C}$, then it certainly satisfies the same condition for $a, b \in \mathbb{R}$ and $z, w \in \mathbb{C}$. But there may be functions in $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ which are not in $\mathcal{L}(\mathbb{C})$. The characterization obtained from $L(z) = zL(1)$ certainly does not work. We can say, however, for $L \in \mathcal{L}(\mathbb{C}_{\mathbb{R}})$ that

$$L(x + yi) = xL(1) + yL(i) \qquad \text{for every } x, y \in \mathbb{R}. \tag{1}$$

Conversely, given any two complex numbers $a_{11} + a_{21}i$ and $a_{12} + a_{22}i$, there is a (unique) linear function $L \in \mathcal{L}(\mathbb{C}_{\mathbb{R}})$ given by

$$L(z) = L(x+iy) = x(a_{11}+a_{21}i)+y(a_{12}+a_{22}i) = xa_{11}+ya_{12}+i(xa_{21}+ya_{22}). \tag{2}$$

Thus, we obtain in this way a characterization of linear functions $L \in \mathcal{L}(\mathbb{C}_{\mathbb{R}})$, and we can ask the question: Can we add and scale such functions?

Certainly we can. If L and M are two such functions, then $L + M : \mathbb{C} \to \mathbb{C}$ obtained by just adding values: $(L + M)(z) = Lz + Mz$ is also the (real) linear function determined by the two complex constants

$$L(1) + M(1) \qquad \text{and} \qquad L(i) + M(i).$$

Also, if L is determined by the pair $(L(1), L(i)) \in \mathbb{C}^2$, then for any real scalar $\alpha \in \mathbb{R}$, the function $\alpha L : \mathbb{C} \to \mathbb{C}$ by $(\alpha L)z = \alpha\, Lz$ corresponds to the pair $\alpha(L(1), L(i))$.

Similarly, all the vector space properties of $\mathcal{L}(\mathbb{C}_\mathbb{R})$ follow from the addition and (real) scaling of points in $\mathbb{C}^2$ which is essentially eqiuvalent to the collection of 2×2 real matrices.

One gets from this discussion the idea that $\mathcal{L}(\mathbb{C}_\mathbb{R})$ is a four dimensional vector space over $\mathbb{R}$ like the 2×2 matrices with real entries. And indeed this is the case: A basis for $\mathcal{L}(\mathbb{C}_\mathbb{R})$ is $\{L_{11}, L_{12}, L_{21}, L_{22}\}$ where

$$L_{11}(z) = \text{Re}(z)$$

corresponding to the two complex constants $1+0i$ and $0+0i$ or... corresponding to the 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The function L_{11} may also be recognized as projection onto the real axis. The second basis element $L_{12} : \mathbb{C} \to \mathbb{C}$ by

$$L_{12}(z) = \text{Im}(z)$$

which is clockwise rotation by $\pi/2$ (multiplication by $-i$) of the projection onto the imaginary axis. The last two basis elements satisfy

$$L_{21}(z) = i\,\text{Re}(z) \qquad \text{and} \qquad L_{22}(z) = i\,\text{Im}(z).$$

The characterization (1-2) gives

$$L = \text{Re}[L(1)]\,L_{11} + \text{Im}[L(1)]\,L_{21} + \text{Re}[L(i)]\,L_{12} + \text{Im}[L(i)]\,L_{22}.$$

Thus, $\{L_{11}, L_{12}, L_{21}, L_{22}\}$ is a spanning set. In order to see $\{L_{11}, L_{12}, L_{21}, L_{22}\}$ is linearly independent and, thus, a basis, consider a linear combination

$$\sum_{i,j} a_{ij} L_{ij} = L_0$$

where $L_0 \equiv 0$ is the zero linear function. Then applying this transformation to $z = 1$ we get

$$a_{11} + a_{21} i = 0.$$

This means $a_{11} = a_{21} = 0$. Applying the remaining terms $a_{12}L_{12} + a_{22}L_{22}$ to i gives

$$a_{12} + i a_{22} = 0,$$

so $a_{12} = a_{22} = 0$. Thus, $\{L_{11}, L_{12}, L_{21}, L_{22}\}$ is a basis for $\mathcal{L}(\mathbb{C}_\mathbb{R})$ over $\mathbb{R}$, and the dimension is 4.

(d) Certainly it is true, as mentioned above, that $\mathcal{L}(\mathbb{C}) \subset \mathcal{L}(\mathbb{C}_{\mathbb{R}})$. However, $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ is a much larger set. In particular, none of the basis elements we have taken for $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ are in $\mathcal{L}(\mathbb{C})$. It may be interesting to express the elements in the basis $\{\text{id}, \rho\}$ for $\mathcal{L}(\mathbb{C})$ as linear combinations of the basis elements in $\{L_{11}, L_{12}, L_{21}, L_{22}\}$. It is easy to see that

$$\text{id} = L_{11} + L_{22} \qquad \text{and} \qquad \rho = L_{21} - L_{12}.$$

Finally, it is perhaps most informative to think of these vector spaces in terms of real linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the Euclidean plane (with which we are relatively familiar) under the identificaiton $x + iy \longleftrightarrow (x, y)$. The linear functions in $\mathcal{L}(\mathbb{C})$ correspond precisely to the compositions of real rotations and real isotropic scalings of $\mathbb{R}^2$ which, it will remembered, commute. The functions in $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ correspond to all linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane.

I feel as though the wording of the problem suggests some additional interesting observation concerning the algebraic properties of $\mathcal{L}(\mathbb{C})$ as a subset of $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$. Ah yes, setting aside the complex scaling associated with $\mathcal{L}(\mathbb{C})$, it may be observed that $\mathcal{L}(\mathbb{C})$ (as a set which is a subset of the real vector space $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$) is a **subspace**. This requires closure under addition, which of course holds, and closure under real scaling, which also holds. So, in a certain sense, there is a final question: What is the (real) dimension of $\mathcal{L}(\mathbb{C})$ as a subspace of $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$? Perhaps I will save that for the final exam.