Exam 1: Solution of Problem 3 And Problem 10

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Problem 1 *Find the power series expansion and radius of convergence with respect to the center of expansion $z = 0$ for the following complex functions:*

(a) *(Boas 14.2.36)* $\sqrt{1 + z^2}$.

(b) *(Boas 14.2.39)* $z/(z^2 + 9)$.

Problem 2 *(Boas 14.2.45) If $f(z) = u + iv$ is a complex differentiable function and $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is the real vector field given by $\mathbf{v} = (v, u)$, then show*

(a) $\operatorname{div} \mathbf{v} = 0$.

(b) $\operatorname{curl} \mathbf{v} = 0$.

Problem 3 *(Boas 14.2.46) Find a system of Cauchy-Riemann equations in polar coordinates for the real and imaginary parts $\xi = \xi(r, \theta)$ and $\eta = \eta(r, \theta)$ of a complex differentiable function $f = \xi + i\eta$. Hint: Write $x = r \cos \theta$ and $y = r \sin \theta$ and apply the chain rule to $u = u(x, y)$ and $v = v(x, y)$.*

Solution: There are two answers one might obtain for this problem, though one is aesthetically rather nicer than the other. Let us begin by noting that the polar coordinates map $\phi : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ by

$$\phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

is 2π periodic in θ and singular at $r = 0$ in the sense that there is no local inverse there and the branches of the local inverse

$$\phi^{-1}(x, y) = (\sqrt{x^2 + y^2}, \operatorname{Arg}(x + iy))$$

become singular there. In particular, the derivative(s) of $r = \sqrt{x^2+y^2}$ do not extend continuously to the origin. Consequently, we will restrict to $r > 0$.

According to the hint we write $\xi(r,\theta) = u(r\cos\theta, r\sin\theta)$ and $\eta(r,\theta) = v(r\cos\theta, r\sin\theta)$ and recall that u and v satisfy the Cauchy-Riemann equations in rectangular coordinates. That is, $u_x = v_y$ and $u_y = -v_x$. Computing using the chain rule we see

$$\xi_r = u_x\cos\theta + u_y\sin\theta = v_y\cos\theta - v_x\sin\theta, \tag{1}$$

$$\xi_\theta = -u_x r\sin\theta + u_y r\cos\theta = -v_y r\sin\theta - v_x r\cos\theta, \tag{2}$$

$$\eta_r = v_x\cos\theta + v_y\sin\theta. \tag{3}$$

$$\eta_\theta = -r v_x\sin\theta + v_y r\cos\theta. \tag{4}$$

Comparing (1) to (4), we observe

$$\xi_r = \frac{1}{r}\,\eta_\theta.$$

Similarly, comparing (3) to (2) gives

$$\xi_\theta = -r\,\eta_r.$$

This is what was intended and is, more or less, what most of you did.

An alternative (still quite correct and equivalent, but not nearly is pretty) is the following: Write $u(x,y) = \xi(\sqrt{x^2+y^2}, \operatorname{Arg}(x+iy))$ and $v(x,y) = \xi(\sqrt{x^2+y^2}, \operatorname{Arg}(x+iy))$, so that

$$u_x = \xi_r\,\frac{x}{r} - \xi_\theta\,\frac{y}{r^2} = \xi_r\,\cos\theta - \xi_\theta\,\frac{\sin\theta}{r},$$

$$u_y = \xi_r\,\frac{y}{r} + \xi_\theta\,\frac{x}{r^2} = \xi_r\,\sin\theta + \xi_\theta\,\frac{\cos\theta}{r},$$

$$v_x = \eta_r\,\frac{x}{r} - \eta_\theta\,\frac{y}{r^2} = \eta_r\,\cos\theta - \eta_\theta\,\frac{\sin\theta}{r},$$

and

$$v_y = \eta_r\,\frac{y}{r} + \eta_\theta\,\frac{x}{r^2} = \eta_r\,\sin\theta + \eta_\theta\,\frac{\cos\theta}{r}.$$

Then we can rewrite the Cauchy-Riemann equations in rectangular coordinates directly as

$$\xi_r\,\cos\theta - \xi_\theta\,\frac{\sin\theta}{r} = \eta_r\,\sin\theta + \eta_\theta\,\frac{\cos\theta}{r}$$

and

$$\xi_r\,\sin\theta + \xi_\theta\,\frac{\cos\theta}{r} = -\eta_r\,\cos\theta + \eta_\theta\,\frac{\sin\theta}{r}.$$

This pair of PDEs for ξ and η may also be considered "The Cauchy-Riemann equations in polar coordinates."

Exercise: Show that if the prettier version of the Cauchy-Riemann equations in polar coordinates hold for ξ and η, then the alternative version also holds. Solve the second set of Cauchy-Riemann equations as a linear system for ξ_r and ξ_θ to get the prettier version. Thus, you will have shown the two are equivalent for $r > 0$.

Problem 4 *(Boas 14.2.48) Express the real and imaginary parts ξ and η of $f(z) = \sqrt{z}$ in polar coordinates, and verify that the Cauchy-Riemann equations in polar coordinates derived in the previous problem hold for ξ and η.*

Problem 5 *(Boas 14.2.47) Find Laplace's equation in polar coordinates.*

Problem 6 *(Boas 14.2.55) Show that $u(x, y) = 3x^2y - y^3$ is harmonic and find a harmonic conjugate for u.*

Problem 7 *Compute*

$$\int_{\partial B_r(z_0)} \frac{1}{z - z_0}.$$

Make the calculation from scratch; do not use the Cauchy integral formula or the residue theorem. What can you say about

$$\lim_{r \searrow 0} \left| \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0} - \int_{\partial B_r(z_0)} \frac{f(z_0)}{z - z_0} \right| ?$$

Hint: Remember $|\int q| \le \int |q|$.

Problem 8 *(Boas 14.3.21) Differentiate the Cauchy integral formula repeatedly (under the integral sign) to obtain a formula for the n-th derivative of a complex differentiable function f.*

Problem 9 *(Boas 14.11.3) Prove Liouville's Theorem: A bounded entire function is constant. The word* **entire** *here means* complex differentiable on all of $\mathbb{C}$*. Hint: Use the previous problem with $n = 1$ and $\Gamma = \partial B_r(z)$, then estimate the integral.*

Problem 10 **(a)** *Use mathematical software to plot the image of the composition $g \circ \gamma(t)$ in the complex plane where $g(z) = e^z$ and $\gamma(t) = re^{it}$ for $0 \le t \le 2\pi$ parameterizes a semicircle for various values of r between 0.1 and 3.*

There is a typo above: The semicircle is given for $0 \le t \le \pi$. Perhaps another "typo" is that I should have suggested that you consider radii greater then 3. (Notice that 3 is just a little less than π; interesting things happen when the radius exceeds π.) The idea was/is that if you really wanted to understand what was going on you would be motivated to use larger radii yourself. In some sense, this problem was presented in "reverse order." That is, the first part is supposed to be motivated by the second part. See the solution below.

(b) *Consider the function* $f : \mathbb{C}\backslash\{\pm i\} \to \mathbb{C}$ *by*

$$f(z) = \frac{e^{iz}}{z^2+1}$$

which has a simple pole at $z = i$. *Integrate around* $\partial B_R^+(0) = \partial\{z \in \mathbb{C} : |z| < R \text{ and } \operatorname{Im}(z) > 0\}$ *and apply the residue theorem to calculate the real integral*

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\, dt.$$

Solution: I'll first plot the images for radii $|z| = 0.1$, 0.5, and 1. To produce the plots below, I parameterized the unit circle by $z = |z|(\cos t + i \sin t) \sim |z|(\cos t, \sin t)$. In rectangular coordinates the value of $w = e^{iz}$ is given by

$$e^{-|z|\sin t}\left(\cos(|z|\cos t), \sin(|z|\cos t)\right). \tag{5}$$

This is the expression you enter into your mathematical software/program. Notice that the modulus of this number is $e^{-|z|\sin t}$ which satisfies

$$0 < e^{-|z|\sin t} \leq 1,$$

so the image will be in the closed unit circle.

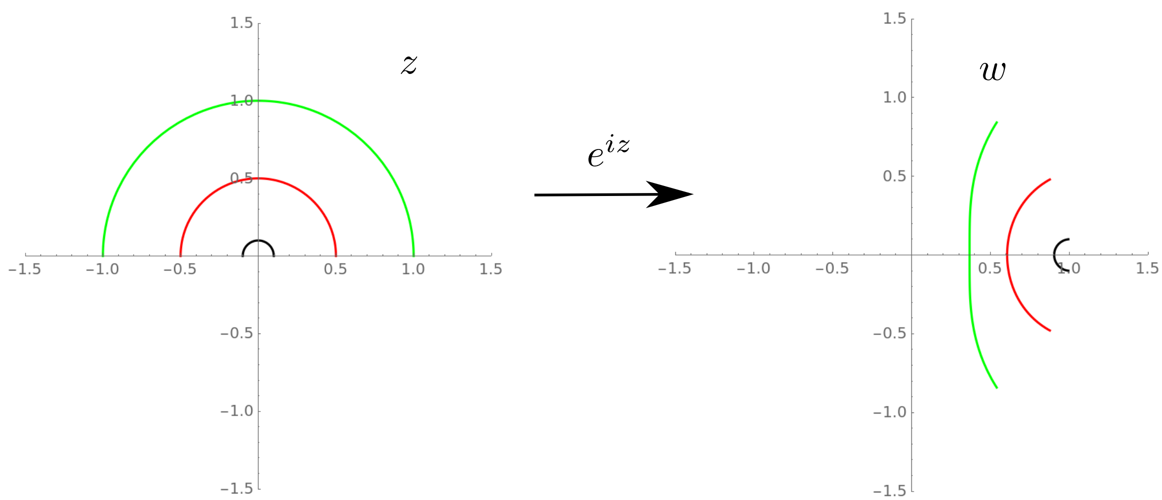

Figure 10.1: *The images of semicircles in the upper half plane under the function* $w = f(z) = e^{iz}$. *Notice how the endpoints of the semicircular arcs "move" away from* $w = 1$ *around the unit circle* $|w| = 1$. *The successive radii* $|z| = 0.1, 0.5, 1$ *are distinguished by the color progression "black, red, green" from smaller to larger.*

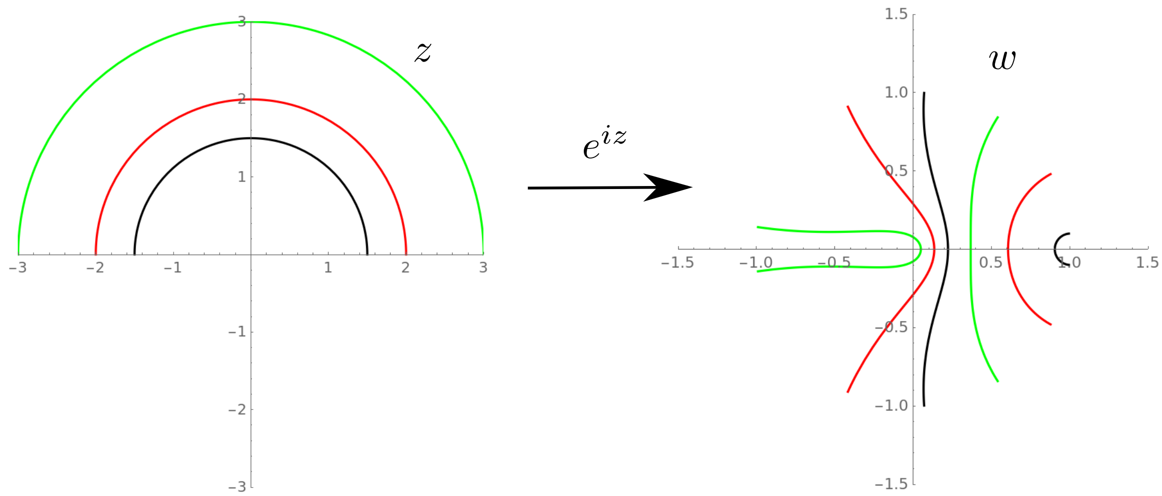

Figure 10.2: *The images of semicircles in the upper half plane under the function* $w = f(z) = e^{iz}$*. I've only drawn the larger circles of radii* $|z| = 1.5, 2, 3$ *in the domain on the left (using the same color progression to distinguish them). In the co-domain on the right, I have kept the images of the first three semicircles from Figure 10.1. The endpoints of the semicircular arcs continue to "move" away from* $w = 1$ *around the unit circle* $|w| = 1$ *when the radius is increased.*

Finally, notice that the scale (a square domain of side length 3*) is maintained in the image* w *plane, but I have zoomed out by a factor of* 2 *in the domain on the left where one sees a square of side length* 6*.*

At this point (it was hoped that) you were wondering what happened when the radii exceeded $|z| = \pi$. Notice 3 is a little less than π, and the image of the endpoints of the semicircular arc with $|z| = 3$ have argument a little less than π. We can guess that if the radius exceeds π, then the image curve should cross itself—unless the image point $w = f(|z|i) = e^{-|z|}$ passes through the origin. Of course, this can't happen because $e^{-|z|} > 0$ is always on the positive real axis in the w plane. It may be noticed that the image $w = e^{-|z|}$ of the half way point $z = |z|i$ is always the point on the image which is closest to the origin.

Zooming out in the domain by another factor of 2, we consider semicircles of radii $|z| = 4$, 5, and 6, and we see that indeed interesting/complicated things start to happen.

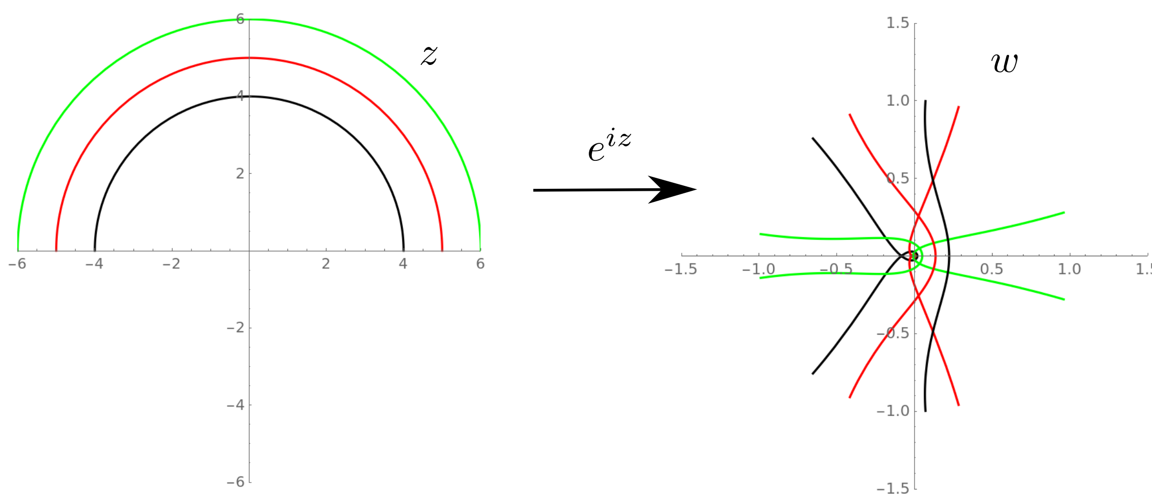

Figure 10.3: *The images of semicircles in the upper half plane under the function $w = f(z) = e^{iz}$. I've only drawn the larger circles of radii $|z| = 4, 5, 6$ in the domain on the left (using the same color progression to distinguish them). In the co-domain on the right, I have kept the images of the second three semicircles from Figure 10.2. The endpoints of the semicircular arcs continue to "move" away from $w = 1$ around the unit circle $|w| = 1$. When the radius is increased to $|z| = 4$ (new black image curve in the w plane on the right) we can see clearly that the curve completes a loop around the origin or winds around the origin once crossing itself. It is not so clear what happens with the images for $|z| = 5$ (new red image curve) and $|z| = 6$ (new green image curve), but the endpoints continue to wind around the unit cicle in opposite directions, and it looks like (maybe) these curves become singular at the origin. Of course, we know this can't happen; we just need to zoom in on the origin and resolve the behavior.*

Notice that the scale with a square of side length 12 in the domain on the left.

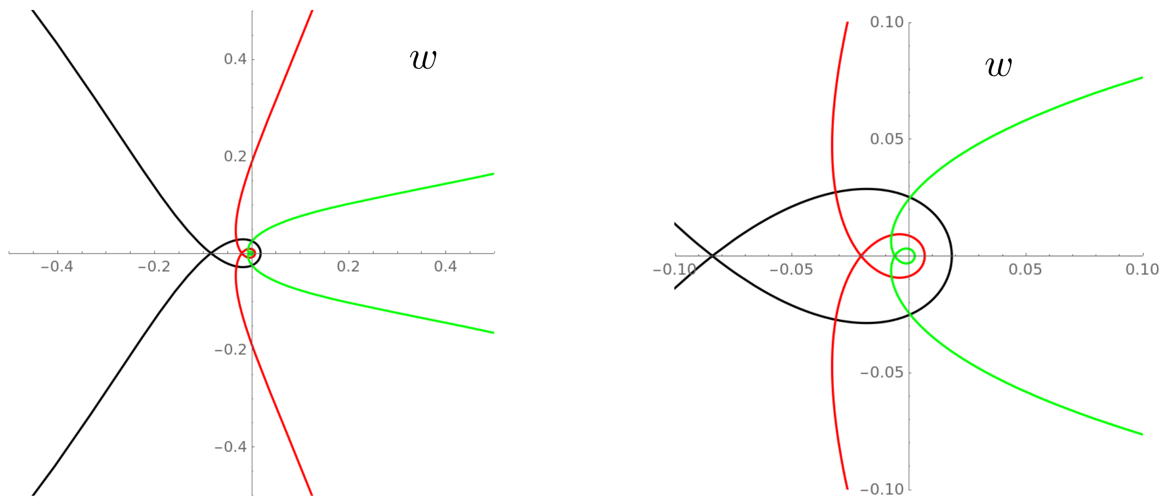

Figure 10.4: *On the left, I have zoomed in on the images of semicircles with radii* $|z| = 4, 5, 6$*. The displayed region is a square domain with side length* 1*. I've omitted the images of all the other (six) semicircles considered above. One can see clearly here that the image of the semicircle with* $|z| = 5$ *(red curve) winds around the origin, is non-singular, and has a single self-intersection. The image of the last semicircle with* $|z| = 6$ *(a bit less than* 2π*) still looks somewhat singular.*

On the right, I have zoomed in to a square of side length 0.2*, and the behavior of all three of these self-intersecting curves near the origin is clear.*

At this point, I think I understand how the (rather complicated) mapping of these semicircles works. To test my understanding, I'm going to plot the image of the semicircle with radius $|z| = 20$. This exercise will illustrate something else unexpected (and instructive) with respect to using mathematical software to understand problems like this.

(The first lesson is this: You may need to zoom in to a different scale to see what is going on. I'm not sure a single student got this valuable lesson when he took the exam. But you have a second chance now. And I'm sure you're excited that another valuable lesson is about to be discussed.)

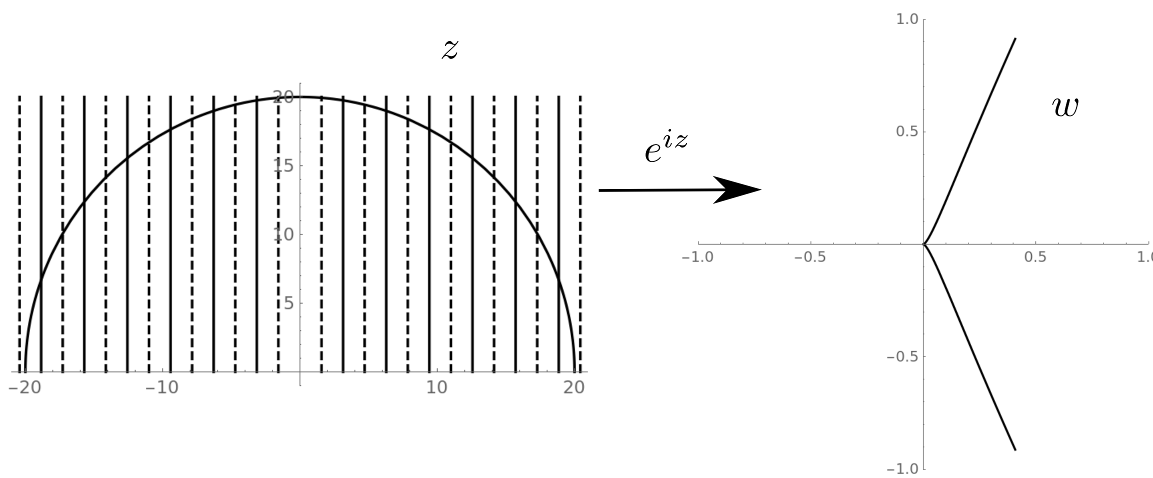

Figure 10.5: *Here we have the domain and image of the semicircle of radius* $|z| = 20$*. I will discuss the expectations concerning the image on the right in detail below. Notice here that at this scale (in a square of side length* 2 *in the* w *plane) the image simply looks singular at the origin.*

As can be seen from the formula (5) above, which we can rewrite in complex notation as

$$f(x + iy) = e^{-y}(\cos x + i \sin x),$$

the argument in the image is determined by the real part of z. Focusing on the domain (on the left in Figure 10.5 where vertical lines at multiples of π are drawn as solid lines and intermediate (dashed) vertical lines at half multiples of π are also drawn) and starting at the endpoint $z = 20$, we observe that the image of this point should have argument between 6π and $13\pi/2$. This is consistent with the endpoint in the unit circle (in the first quadrant) in the image on the right. Moving around the semicircle in the counterclockwise direction, the real part x decreases corresponding to a decreasing argument for w in the co-domain, and this is what we see.

What we should see (again focusing on the domain) is that the argument of w continues to decrease as the image curve winds around the origin (in the clockwise direction) a total of three times before reaching the half-way point $w = f(20i) = e^{-20}$ closest to the origin and on the real axis. Continuing along the quarter circle $|z| = 20$ in the second quadrant, the image should wind back out determining a total of six intersection points on the real axis of progressively greater modulus (and corresponding to $x = |z| \cos t$ taking the values $-\pi$, -2π, -3π, -4π, -5π, and -6π. Three of these intersection points (the ones corresponding to odd multiples of π) should be on the negative real axis, and the other three (corresponding to the even multiples of π) should be on the positive real axis. All of this should be happening in what appears to be a singular point on the image curve on the right in the w plane of Figure 10.5. Let's see if we can verify/resolve this behavior.

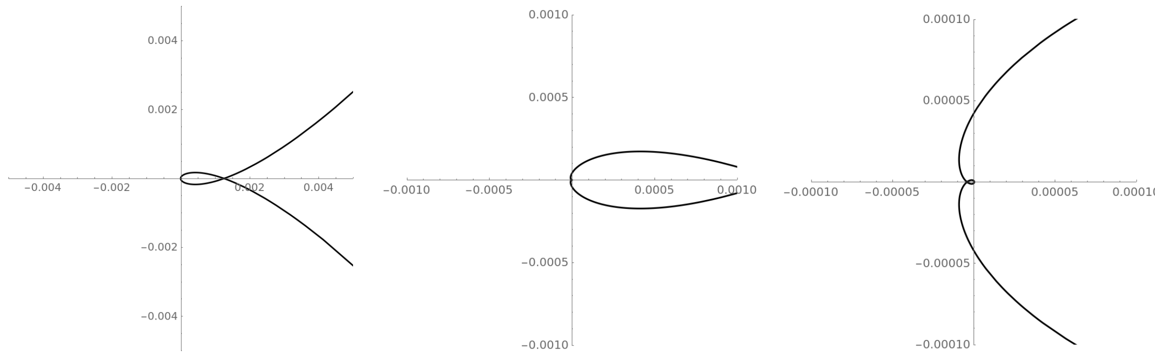

Figure 10.6: *In a square of side length* 0.01 *in the* w *plane, we pick up the last of the six intersection points according to the description above. (left) Zooming in to a square of side length* 0.002 *(20× magnification) we see what appears to be a loop with a hint of singular behavior at the origin. (middle) Zooming in by another factor of* 10*, we see the second to last intersection point on the negative real axis. (The entire plot on the right shows the "tip" of the "loop" in the middle and left plots.)*

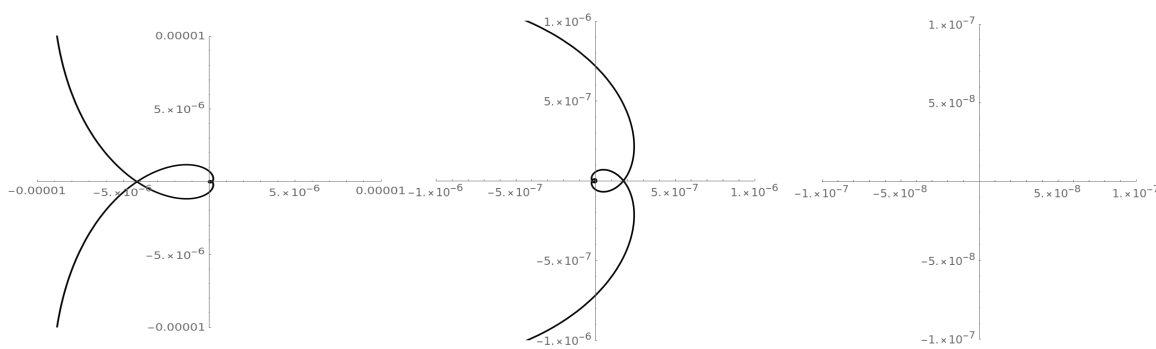

Figure 10.7: *The plot on the left is a* 10× *magnification of the "loop" seen on the right in Figure 10.6. This shows clearly self-intersection point number* 5*, and one can almost make out self-intersection point number* 4 *on the positive real axis. In the middle we have zoomed in by another factor of 10, and intersection point number* 4 *is clearly visible. Then something unexpected happens: When we zoom in by another factor of* 10*, the entire image disappears. Nothing is visible.*

One might initially imagine we have zoomed in too much and our viewing window is so far inside the innermost loop/wind that nothing is visible. This can certainly happen, but a little contemplation of the scale indicates that this is not what has happened here. Something else has gone wrong. The problem is almost certainly related to a lack of resulution in the parameter domain for the computation. Roughly speaking, when Mathematica plots a curve like this, a certain finite number of points, say between $t = 0$ and $t = \pi$ are chosen. When we continue to zoom in on the image around $t = \pi/2$, very few of those points are included in what we see. When the number of displayed points becomes too small in the viewing window, Mathematica gets suspicious (so to speak) and won't display anything. At any rate, the problem is fixed by replotting on a smaller interval, say $\pi/2-1 \le t \le \pi/2+1$, and then zooming in.

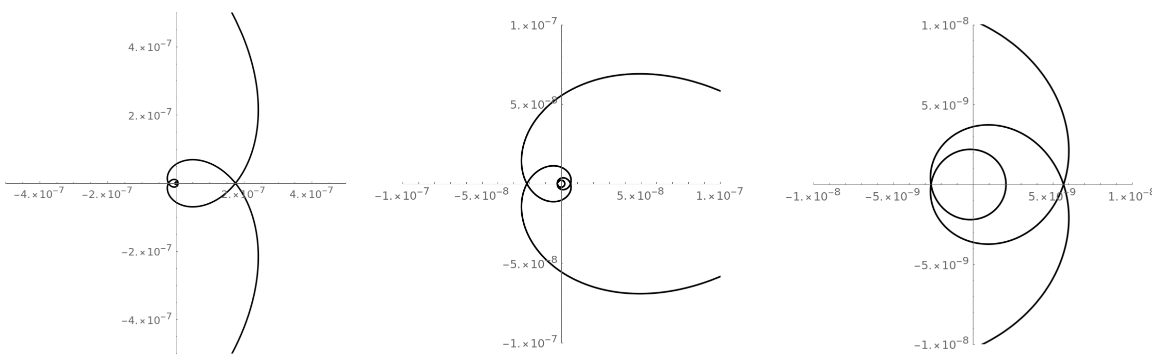

Figure 10.8: *Here we pick up on the left with a* $2\times$ *magnification of the middle plot in Figure 10.7 (around where we lost output). In the middle, we see clearly intersection point number* 3 *on the negative real axis. Zooming in by another factor of* 10 *on the right, we finally see the first self-intersection point on the negative real axis, the second self-intersection point on the positive real axis, and the point closest to the origin (half-way point corresponding to* $t = \pi/2$*) on the positive real axis.*

Everything is as predicted.

Exercise: Go backwards through Figures 10.8 through 10.5 starting at the half-way point corresponding to $t = \pi/2$. Continue along the image curve in a counterclockwise direction from plot to plot (zooming out) and labeling each self-intersection point as it is crossed.

Exercise: At what magnification do you find nothing is visible because the viewing window is entirely inside the inner loop on the right in Figure 10.8?

Okay, I think we are ready to do part (b) and discuss the point of part (a). The semicircle from part (a) is part of the countour Γ around which we are supposed to integrate. The other part is a straight line segment on the real axis between $-R$ and R. In coordinates, we get

$$\int_\Gamma \frac{e^{iz}}{z^2+1} = \int_0^\pi \frac{e^{-R\sin t}\, e^{iR\cos t}}{R^2e^{2it}+1}\, iRe^{it}\, dt + \int_{-R}^{R} \frac{e^{it}}{t^2+1}\, dt = 2\pi i \operatorname{res}_i\left(\frac{e^{iz}}{z^2+1}\right).$$

Notice we have used the parameterization $z = \gamma(t) = Re^{it} = R\cos t + iR\sin t$ in the first integral, and the associated real derivative is

$$\frac{dz}{dt} = \gamma'(t) = iRe^{it}.$$

In the second integral we have used $z = \gamma(t) = t$, and this second integral limits, as R tends to $+\infty$, to

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\, dt$$

which is precisely the integral we are asked to compute in part (b). There is also a term

$$i\int_{-R}^{R} \frac{\sin t}{t^2+1}\, dt$$

in the second integral, but you will note that since the integrand it odd and the interval of integration is symmetric, this vanishes.

There are two things we need to do now. One is to compute the residue. Remember that (in this case) the residue is the coefficient of the term $(z-i)^{-1}$ in the Laurent expansion. The functions e^{iz} and $(z+i)^{-1}$ are analytic near $z = i$, so this is pretty easy:

$$\begin{aligned}\frac{e^{iz}}{z^2+1} &= e^{-1}\, e^{i(z-i)}\, \frac{1}{z+i}\, \frac{1}{z-i} \\ &= \frac{1}{e}\, \left(1 + i(z-i) + \mathbf{O}(z-i)^2\right) \left(\frac{1}{2i} - \frac{1}{(2i)^2}(z-i) + \mathbf{O}(z-i)^2\right) \frac{1}{z-i}\end{aligned}$$

where $\mathbf{O}(z-i)^2$ means "more terms involving (quadratic and) higher powers of $z-i$." Thus,

$$\frac{e^{iz}}{z^2+1} = -\frac{i}{2e}\, \frac{1}{z-i} + g(z)$$

where g is analytic, and

$$\operatorname{res}_i\left(\frac{e^{iz}}{z^2+1}\right) = -\frac{i}{2e}.$$

The second thing we need to do is figure out what to do with the integral around the semicircle. As we know from part (a), this integrand is rather complicated for large R, so things don't look particularly good. We had better hope the limit is zero. If that is true, we need an estimate. That $e^{-R\sin t}$ term looks rather helpful, however, when t is close to 0 and π (the endpoints of the semicircle), then this term has modulus 1, so we need to be somewhat careful if we're going to do this right.

We start by taking the absolute value/modulus of the integral:

$$\left|\int_0^\pi \frac{e^{-R\sin t}\, e^{iR\cos t}}{R^2e^{2it}+1}\, iRe^{it}\, dt\right| \le \int_0^\pi \frac{Re^{-R\sin t}}{|R^2e^{2it}+1|}\, dt. \tag{6}$$

Furthermore, $|R^2e^{2it}+1| \ge |R^2e^{2it}| - 1 = R^2 - 1$ by the triangle inequality. In particular, if $R > 3$, then $R^2 - 1 > 2R$ and

$$\left|\int_0^\pi \frac{e^{-R\sin t}\, e^{iR\cos t}}{R^2e^{2it}+1}\, iRe^{it}\, dt\right| \le \frac{1}{2}\int_0^\pi e^{-R\sin t}\, dt.$$

This looks a lot better.

If you think about it (plot $e^{R\sin t}$ for $0 \le t \le \pi$ for several values of R) the only way we can get a limit of zero is if the length of the interval of integration where $e^{-R\sin t}$ is not close to zero is very small. This means we need to split up the interval of integration. To simplify this splitting up of the integral, let's first reduce to an interval over $0 \le t \le \pi/2$ using the change of variable $\tau = \pi - t$ (so we only have one tricky end with which to deal).

$$\int_0^\pi e^{-R\sin t}\, dt = \int_0^{\pi/2} e^{-R\sin t}\, dt - \int_{\pi/2}^0 e^{-R\sin\tau}\, d\tau = 2\int_0^{\pi/2} e^{-R\sin t}\, dt.$$

This means, of course, that

$$\left|\int_0^\pi \frac{e^{-R\sin t}\, e^{iR\cos t}}{R^2e^{2it}+1}\, iRe^{it}\, dt\right| \le \int_0^{\pi/2} e^{-R\sin t}\, dt,$$

and our task is reduced to showing

$$\lim_{R\nearrow\infty} \int_0^{\pi/2} e^{-R\sin t}\, dt = 0.$$

Let's see if we can do it. Let a be a number with $0 < a < \pi/2$.

$$\int_0^{\pi/2} e^{-R\sin t}\, dt = \int_0^a e^{-R\sin t}\, dt + \int_a^{\pi/2} e^{-R\sin t}\, dt \le a + e^{-R\sin a}(\pi/2 - a) \le a + \frac{\pi e^{-R\sin a}}{2}.$$

We need $a = a(R)$ to tend to 0 with R but it should go there slowly enough so that $e^{-R\sin a}$ still tends to zero. The choice $a(R) = 1/\sqrt{R}$ should do the job since

$$\lim_{R\nearrow\infty} R\sin(1/\sqrt{R}) = \lim_{R\nearrow\infty} \frac{\cos(1/\sqrt{R})}{2R^{3/2}/R^2} = +\infty.$$

Exercise: Go back to (6) and obtain estimates giving the same result based on the better estimate $R^2 - 1 > R^2/2$ for $R > \sqrt{2}$.

Taking the limit as R tends to ∞ and substituting the value of the residue calculated above, we get

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\, dt = \frac{\pi}{e}.$$

I hope I have given a, more or less, correct solution. And maybe you can look back and see now that this integral around the semicircle, which we estimated and limited out of our hair, is (for all its limiting to zero) quite a complicated quantity.

(Bonus) At the end of my posted solution for Assignment 2 Problem 1, I mentioned several guesses (or conjectures) concerning the values of the alternating harmonic series

$$f(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1}.$$

I wrote there "I have no idea how to prove these assertions."

Find the function f explicitly.

Hints:

(a) Think carefully about Exercise 7 in my solution to Problem 1 of Assignment 2.

(b) Review Chapter 2 Sections 6 and 7 of Boas.

(c) Use (your solution of) Problem 10 of Assignment 1 to see geometrically precisely what is going on and get the formula.

When you are done you should be able to produce/reproduce Figure 4 of my solution to Problem 1 of Assignment 2 with relative ease.