Final Assignment: Cumulative/review
*** under construction ***
Due Thursday December 17, 2026

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Problem 1 Recall the cyclic group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ containing four elements. Find another group with four elements (which is not isomorphic to $\mathbb{Z}_2$).

Problem 2 Assume $a_1, a_2, a_3, \ldots$ is a sequence of real numbers for which

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

That is, the series

$$\sum_{n=1}^{\infty} a_n$$

of real numbers is absolutely convergent. Follow the steps below to show the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent to a well-defined real number.

(a) Denote by σ_K the partial sum

$$\sigma_K = \sum_{n=1}^{K-1} a_n.$$

Show there exist real numbers λ_0 and μ_0 for which

$$\lambda_0 \le \sigma_K \le \mu_0 \qquad \text{for every } K = 1, 2, 3, \ldots .0$$

That is, the sequence of partial sums is bounded with lower bound λ_0 and upper bound μ_0. Hint: For $K > N$ write

$$\sigma_K = \sigma_N + \sum_{n=N}^{K-1} a_n$$

and use the triangle inequality.

(b) For $N = 1, 2, 3, \ldots$ let

$$\lambda_N = \inf\{\sigma_K : K \geq N\}$$

where the inf (infemum) of a nonempty set of real numbers which is bounded below is the **greatest lower bound** of that set. (It is an important analytic property of the real numbers that every nonempty set A of real numbers which is bounded below has a well-defined greatest lower bound $\inf A$. This is another way of saying the real number line has no "gaps" or "holes.")

Show $\{\lambda_N\}_{N=1}^{\infty}$ is a nondecreasing sequence of real numbers which is bounded above.

(c) Show there is a nonincreasing sequence of real numbers $\mu_1, \mu_2, \mu_3, \ldots$ satisfying

$$\lambda_N \leq \sigma_N \leq \mu_N$$

for $N = 1, 2, 3, \ldots$ and

$$\lim_{N\to\infty} \mu_N - \lambda_N = 0.$$

Hint: Given a set $A \subset \mathbb{R}$ with A bounded above, there is a unique real number which is the **least upper bound** of A denoted by $\sup A$ and called the **supremum**.

(d) Show

$$L = \lim_{N\to\infty} \lambda_N = \lim_{N\to\infty} \mu_N$$

and conclude

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sigma_N$$

exists as a well-defined real number.