Assignment 9: Complex Analysis
Due Thursday December 3, 2026

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As usual given $r > 0$ and $z_0 \in \mathbb{C}$

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Careful consideration of Problem 10 of Assignment 8 leads to the following result:

Lemma 1 (Ahlfors' lemma) Given $\phi \in C^0(\partial B_r(z_0) \to \mathbb{C})$ and $n \in \mathbb{N}_0$, the function $F_n : B_r(z_0) \to \mathbb{C}$ by

$$F_n(z) = \int_{\zeta \in \partial B_r(z_0)} \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}$$

is complex differentiable at each point $z \in B_r(z_0)$ with

$$F_n'(z) = (n+1)F_{n+1}(z).$$

Problem 1 Let $f : U \to \mathbb{C}$ be a complex differentiable function on an open set $U \subset \mathbb{C}$. Assume $\overline{B_r(z_0)} \subset U$. Use Ahlfor's lemma to show the n-th complex derivative of f for $z \in B_r(z_0)$ is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\zeta \in \partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

Hint: Start with Cauchy's integral formula.

Theorem 1 (Cauchy's integral formula) If $f : U \to \mathbb{C}$ is complex differentiable on an open set $U \subset \mathbb{C}$ and $\Omega \subset\subset U$ with $\Gamma = \partial\Omega \subset U$ a C^1 simple closed curve, then

$$f(z) = \int_{\zeta \in \Gamma} \frac{f(\zeta)}{\zeta - z} \qquad \text{for every } z \in \Omega. \tag{1}$$

Problem 2 Give an example of a complex differentiable function $f : U \to \mathbb{C}$ and open sets Ω and U for which $\Omega \subset U$ and $\Gamma = \partial\Omega \subset U$ but (1) does not hold.

Problem 3 Let $f : U \to \mathbb{C}$ and Ω satisfy the hypotheses of Cauchy's integral formula. In this case $\mathbb{C}\backslash\Gamma = \Omega_{\text{in}} \cup \Omega_{\text{out}}$ where Ω_{in} and Ω_{out} are uniquely determined open subsets of $\mathbb{C}$ with Ω_{in} bounded. In fact, $\Omega = \Omega_{\text{in}}$. (This is a consequence of the Jordan curve theorem.) What is the value of the integral

$$\int_{\zeta \in \Gamma} \frac{f(\zeta)}{\zeta - z}$$

for $z \in U \cap \Omega_{\text{out}}$? (Explain the reason for your answer.)