Assignment 8: Complex Integration
Due Thursday November 19, 2026

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Definition 1 A **complex contour** is a curve $\Gamma \subset \mathbb{C}$ with a specified orientation.

This definition is somewhat vague on the regularity of the curve and the meaning of orientation. These two related aspects of the definition can be difficult to nail down carefully. Notice the definition says nothing about a parameterization, but if one has a parameterization, then the details about contours are easy to understand. Specifically, if $\alpha : I \to \mathbb{C}$ has image $\alpha(I) = \Gamma$ where $I = [a, b]$ is a closed interval in $\mathbb{R}$, then the regularity of Γ can be identified with the regularity of α. Recall $\alpha \in C^1(I \to \mathbb{C})$ if $\alpha = \xi + i\eta$ with $\xi = \mathrm{Re}(\alpha)$ and $\eta = \mathrm{Im}(\alpha)$ satisfying $\xi, \eta \in C^1(I)$. Such a parameterization is said to be **regular** if $|\alpha'(t)| = |\xi' + i\eta'| \neq 0$ for all $t \in I$. Remember this derivative $\alpha' = \xi' + i\eta'$ is not a complex derivative but rather a real derivative of a complex valued function (of one variable). All contours considered below at least admit a parameterization $\alpha \in C^0(I \to \Gamma)$ which is piecewise C^1 as well. (See the definition of partition below.)

If $\alpha(t_1) \neq \alpha(t_2)$ for $t_1, t_2 \in I$ with $t_1 < t_2$, then $\Gamma = \alpha(I)$ is said to be **simple**. The orientation of a simple curve is easy to understand: $\alpha(a)$ and $\alpha(b)$ are distinct points, and there are two orientations corresponding to two directions on the curve

either from $\alpha(a)$ *to* $\alpha(b)$ **or**...
from $\alpha(b)$ *to* $\alpha(a)$.

Unfortunately not all curves considered below admit a simple and/or regular parameterization, but hopefully you can guess (or figure out) how more complicated contours work as you go through the problems. One special case deserves mention: If Γ is a contour and one can find a parameterization $\alpha : I \to \mathbb{C}$ with $\alpha(I) = \Gamma$ and

1. $\alpha(b) = \alpha(a)$, but
2. $\{(t_1, t_2) \in I^2 : t_1 < t_2 \text{ and } \alpha(t_1) = \alpha(t_2)\} = \{(a, b)\}$,

then Γ is said to be a **cycle** or **cyclic contour** or **simple closed curve**. The orientation of a simple cycle in $\mathbb{C}$ is usually specified as **clockwise** or **counterclockwise**. If there is no specification for a cycle, a counterclockwise orientation is usually assumed.

If $\alpha \in C^1(I \to \mathbb{C})$ is a regular parameterization, then the **speed** of the parameterization at $t \in I$ is $|\alpha'(t)|$.

Problem 1 Given $a, b > 0$ and $w \in \mathbb{C}$ parameterize the circular contour

$$\partial B_a(w) = \{z \in \mathbb{C} : |z - w| = a\}$$

in a counterclockwise direction with (constant) speed b. Be sure to specify the domain of your parameterization(s).

Problem 2 Given $z_0, z_1 \in \mathbb{C}$ with $z_0 \neq z_1$ and $\beta > 0$, parameterize the line segment connecting z_0 to z_1 in the direction from z_0 to z_1 with constant speed β.

Definition 2 A **partition** $\mathcal{P}$ of a contour Γ is a (finite) sequence

$$\mathcal{P} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$$

of contours Γ_j compatible with the orientation of Γ in the sense that Γ_{j+1} comes after Γ_j and satisfying the following:

1. The union of the partition contour is the entire contour Γ:

$$\bigcup_{j=1}^{n} \Gamma_j = \Gamma.$$

2. The intersection of consecutive partition contours give a sequence of consecutive points:

$$\Gamma_{j+1} \cap \Gamma_j = \{z_j\}.$$

This holds in general for $j = 1, 2, \ldots, n-1$. For a cycle one may also include $\Gamma_n \cap \Gamma_1 = \{z_0\} = \{z_n\}$. For a non-cycle, one may wish to include an initial endpoint z_0 of Γ_1 and a final endpoint z_n of Γ_n in the sequence.

Note the endpoints of Γ_j are z_j and z_{j-1}.

A parameterization $\alpha : I \to \Gamma$ of a contour is said to be **piecewise** C^1 if there exist partitions $\{I_1, I_2, \ldots, I_n\}$ of I and $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ of Γ for which $\alpha(I_j) = \Gamma_j$ and

$$\alpha\big|_{I_j} \in C^1(I_j \to \mathbb{C}).$$

Given a contour Γ and a function $f \in C^0(\Gamma \to \mathbb{C})$, a **complex integral** (when it exists) is the (complex) number

$$\int_\Gamma f = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} f(z_j^*)(z_j - z_{j-1}) \tag{1}$$

where

$$\|\mathcal{P}\| = \max\{\text{diam}(\Gamma_j) : j = 1, 2, \ldots, n\} \qquad \text{and} \qquad \text{diam}(A) = \sup\{|z-w| : z, w \in A\},$$

$z_j^* \in \Gamma_j$ is an evaluation point, and the limit is taken over all possible partitions and evaluation points.

The complex integral may be considered to have two arguments f and Γ. In particular, if $\mathcal{G}$ denotes the collection of all complex contours and $\mathcal{F}$ denotes the collection of all complex valued functions, then the integral

$$I(f, \Gamma) = \int_\Gamma f$$

defines a functional

$$I : \{(f, \Gamma) \in \mathcal{F} \times \mathcal{G} : f \in C^0(\Gamma \to \mathbb{C})\} \to \mathbb{C}.$$

The fact that a contour is **oriented** is somewhat special to complex integration, though other curves (for example in $\mathbb{R}^n$) are also sometimes considered with an orientation. Having an orientation does not require the curve to have a specific parameterization but only a specific direction.

Problem 3 Though it is not the definition of the complex integral (1) the value of such an integral is usually (or at least often) calculated using the formula

$$\int_\Gamma f = \int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt \tag{2}$$

where $\alpha \in C^1([a, b] \to \mathbb{C})$ is a parameterization of Γ compatible wtih the orientation of Γ. The integral on the right in (2) is not a complex integral but rather expresses the value of a complex integral. The integral on the right in (2) is in fact a real integral of a complex valued function

$$\int_a^b [u(t) + iv(t)]\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

with $u, v \in C^1[a,b]$. Again $\alpha' = \xi' + i\eta'$ is a real derivative of a complex valued function $\alpha = \xi + i\eta$ with $\xi, \eta \in C^1[a,b]$.

Explain why formula (2) is at least compatible with the definition (1).

Problem 4 Given $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z$, use formula (2) to compute

$$\int_\Gamma f$$

when

(a) Γ is the line segment from z_0 to z_1.

(b) Γ is the rectangular contour with corners $a+bi$, $c+di$, $c+bi$ and $a+di$ (parameterized counterclockwise). You may assume $a < c$ and $b < d$.

Do your answers depend on the parameterization?

A "variable of integration" is often included in the notation for a complex integral

$$\int_\Gamma f$$

which is sometimes written

$$\int_\Gamma f(z) \qquad \text{or even} \qquad \int_\Gamma f(z)\,dz$$

with a value of f (given by a formula) appearing in place of the name of the function f. Notice this notation mimics the notation

$$\int_a^b u(t)\,dt$$

for real integrals. I will try to avoid using this notation for complex integrals in general, but using such formulas is sometimes convenient. The variation I will sometimes use is

$$\int_{\zeta\in\Gamma} f(\zeta).$$

Here ζ is the variable of integration.

Problem 5 Given $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $f(z) = 1/z$, use formula (2) to compute

$$\int_\Gamma f \qquad \text{or} \qquad \int_{\zeta\in\Gamma} \frac{1}{\zeta}$$

when

(a) Γ is the rectangular contour with corners $\pm a \pm bi$ with $a, b > 0$ and Γ oriented counterclockwise.

(b) Γ is the circle $\partial B_a(z_0) \subset \mathbb{C}$.

Hint(s): In part **(a)** choose a branch of the complex logarithm with a particular side of the rectangular contour in the interior of the domain sheet. See Problems 5 and 8 of Assignment 6.

Problem 6 Given $\Gamma = \partial B_r(0) \subset \mathbb{C}$, use formula (2) to compute

$$\int_\Gamma f$$

when

(a) $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $f(z) = 1/z^2$.

(b) $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $f(z) = 1/z^n$, $n = 3, 4, \ldots$.

A theorem about real integrals

Theorem 1 (divergence theorem) Given a bounded open set $U \subset \mathbb{R}^n$ and a real vector field

$$\mathbf{v} \in C^1(U \to \mathbb{R}^n),$$

meaning $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ with

$$\frac{\partial v_j}{\partial x_k} \in C^0(U)$$

for $j, k = 1, 2, \ldots, n$,

$$\int_{\partial\Omega} \mathbf{v} \cdot N = \int_{\Omega} \operatorname{div} \mathbf{v} \tag{3}$$

for every subdomain Ω nice enough for the integrals to make sense and nice enough to determine an outward unit normal N (at most points of $\partial\Omega$).

For the applications below the dimension is $n = 2$ matching the real dimension of the complex plane $\mathbb{C}$. This means $\partial\Omega$ is typically a curve, and one can take a definition of the real integrals

$$\int_{\Omega} g = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} g(p_j^*) \operatorname{area}(\Omega_j)$$

and

$$\int_{\partial\Omega} g = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} g(p_j^*) \operatorname{length}(\Gamma_j) \tag{4}$$

fairly similar to the definition of the complex integral where in the first case $\{\Omega_j\}_{j=1}^n$ is a partition of Ω and $g \in C^0(\overline{\Omega})$ and in the second case $\{\Gamma_j\}_{j=1}^n$ is a partition of $\Gamma = \partial\Omega$ and $g \in C^0(\partial\Omega)$. No details of orientation are necessary here. In particular, the outward normal takes care of any orientation questions on $\partial\Omega$.

There are various evaluation formulas for the values of these integrals, but I will leave those aside for the moment unless they are needed. (Perhaps you can figure such formulas out yourself.) For the sake of the immediate use below, one can view the divergence $\operatorname{div} \mathbf{v}$ as the real valued function $\operatorname{div} \mathbf{v} : U \to \mathbb{R}$ with values given by

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}(\mathbf{x}).$$

There is a more general definition of the divergence which should be covered next semester along with an explanation of why the divergence theorem holds.

Problem 7 (Cauchy's theorem; Boas section 14.3)

(a) (Green's theorem) Let Ω be a bounded open subset of $\mathbb{R}^2$ which is *simply connected*. Assume further $\partial\Omega$ is a connected curve for which the notion of being oriented counterclockwise makes sense. A disk $B_r(\mathbf{x})$ and a rectangular domain are examples. Such a domain admits an outward unit normal N and a counterclockwise unit tangent related by

$$T = (T_1, T_2) = N^\perp = (-N_2, N_1).$$

Use the divergence theorem to show that under the assumptions of the divergence theorem

$$\int_{\partial\Omega} \mathbf{v} \cdot T = \int_\Omega \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

Hint: Rotate both $\mathbf{v}$ and T clockwise.

(b) Recall that if $f : U \to \mathbb{C}$ is complex differentiable on an open subset $U \subset \mathbb{C}$, then there is a natural way to identify U with a domain $\check{U} \subset \mathbb{R}^2$ (using $x+iy \mapsto (x, y)$). Recall also the standard real and imaginary parts of $f = u+iv$ satisfying $u, v \in C^1(\check{U})$ and the Cauchy-Riemann equations.

(i) Write an expression for f' in terms of

$$\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial v}{\partial x}.$$

(ii) Write an expression for f' in terms of

$$\frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial y}.$$

(c) If Ω is a simply connected domain with $\Omega \subset\subset U$, meaning $\overline{\Omega} \subset U$, then (show)

$$\int_{\partial\Omega} f = 0. \tag{5}$$

Here you probably will need a formula for the value of the boundary integral (4):

$$\int_{\partial\check{\Omega}} g = \int_a^b g \circ \check{\alpha}(t)\ \check{\alpha}'(t)\, dt$$

where $\check{\alpha} : I = [a, b] \to \partial\check{\Omega} \subset \mathbb{R}^2$ by

$$\check{\alpha}(t) = (\alpha_1(t), \alpha_2(t))$$

and $\alpha : I \to \mathbb{C}$ is a parameterization of $\partial\Omega \subset \mathbb{C}$.

Hint(s): As the notation above suggests, assume you can parameterize $\partial\Omega \subset \mathbb{C}$ on some interval $I = (a, b)$ by a function $\alpha = \xi + i\eta$. Express

$$\int_{\partial\Omega} f$$

using formula (2). Then express the real and imaginary parts of the integral as real integrals involving $u \circ \check{\alpha}$ and $v \circ \check{\alpha}$. Finally, use the divergence theorem (or Green's theorem) along with the Cauchy-Riemann equations.

Problem 8 (Cauchy's integral formula) Assume $f : U \to \mathbb{C}$ is complex differentiable.

(a) Assume $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ and $\overline{B_{r_2}(w)} \subset U$. Compute

$$\int_{\zeta \in \partial B_{r_2}(w)} \frac{f(\zeta)}{\zeta - w} - \int_{\zeta \in \partial B_{r_1}(w)} \frac{f(\zeta)}{\zeta - w}.$$

Hint(s): Consider the contour Γ parameterized by $\alpha : [r_1, 2(r_2+2(\pi-\delta))-r_1] \to \mathbb{C}$ with

$$\alpha(t) = \begin{cases} te^{i\delta}, & r_1 \le t \le r_2 \\ r_2 e^{i(t-r_2+\delta)}, & r_2 \le t \le r_2 + 2(\pi - \delta) \\ (2(r_2+\pi-\delta) - t)e^{-i\delta}, & r_2 + 2(\pi-\delta) \le t \\ & \qquad \le 2(r_2+\pi-\delta) - r_1 \\ r_1 e^{i[2(r_2+\pi)-3\delta-r_1-t]}, & 2(r_2+\pi-\delta) - r_1 \le t \\ & \qquad \le 2(r_2+2(\pi-\delta)) - r_1 \end{cases}$$

where $0 < \delta < \pi$. Use Cauchy's theorem (Problem 7) to evaluate

$$\int_\Gamma g$$

where

$$g(\zeta) = \frac{\zeta}{\zeta - w}.$$

Specify an appropriate domain for the complex differentiable function g. Take the limit as $\delta \searrow 0$.

(b) Assume $r < 0$ and $\overline{B_r(w)} \subset U$. Show

$$f(w) = \frac{1}{2\pi i} \int_{\zeta \in \partial B_r(w)} \frac{f(\zeta)}{\zeta - w}. \tag{6}$$

Hint(s): Use part **(a)** with $r_2 = r$, and take the limit as $r_1 \searrow 0$.

Problem 9 (a complex differentiable function; Ahlfors' lemma) Assume $\Gamma = \partial\Omega$ is a simple closed contour in $\mathbb{C}$ and $\phi \in C^0(\Gamma)$. Consider $F : \Omega \to \mathbb{C}$ by

$$F(z) = \int_{\zeta\in\Gamma} \frac{\phi(\zeta)}{\zeta - z}.$$

(a) For $z_0, z \in \Omega$ derive the formula

$$\frac{F(w) - F(z)}{w - z} = \int_{\zeta\in\Gamma} \frac{\phi(\zeta)}{(\zeta - w)(\zeta - z)}$$

for the difference quotient.

(b) Show F is complex differentiable with

$$F'(z) = \int_{\zeta\in\Gamma} \frac{\phi(\zeta)}{(\zeta - z)^2}.$$

(b) What does part **(a)** tell you about the general complex differentiable function in the Cauchy integral formula (6)? Hint: What do you get by differentiating under the integral in (6)?

Problem 10 Compute

$$\int_{\zeta\in\partial B_2(0)} \frac{1}{\zeta^2 + 1}.$$

Hint: Use partial fractions.