Assignment 7: Linear theory of ODEs and higher order equations *** Corrected Version *** Due Thursday November 12, 2026

John McCuan

Definition 1 The **standard inner product** $\langle \cdot\,,\,\cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ on $\mathbb{C}^n$ is given by

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

where $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$.

Two vectors $z, w \in \mathbb{C}^n$ are said to be **orthogonal** if $\langle z, w \rangle = 0$.

A set of vectors $\{v_1, v_2, \ldots, v_k\} \subset \mathbb{C}^n$ is said to be **orthonormal** if

$$\langle v_j, v_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

Problem 1 Consider $L : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$L(z_1, z_2) = (-z_1, z_2).$$

(a) Find an orthonormal basis for $\mathbb{C}^2$ consisting of eigenvectors for L.

(b) Find all solutions $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$ of the linear system $\gamma' = L(\gamma)$.

Note: The ODE considered in Problem 2 below was (sort of) not the ODE I intended to have as the subject of Problems 2-6 below. Actually, the situation is a little more complicated. In fact, I started with the ODE $\gamma' = (-\gamma_1, \gamma_2)$ as in Problem 1 above. I was going to give that one as a really easy (diagonalized/decoupled) system of complex ODEs just sort of as a warm-up. Then I was going to transition to the real ODE I wanted for most of the assignment, namely,

$$\gamma' = (-\gamma_2, \gamma_1)$$

which is both complex valued and coupled. But by the time I finished composing the assignment, I decided I didn't have enough problems to include the warm-up, so I needed to go back change the system. I thought I did that for Problems 2-6, but obviously I had the wrong formula for the linear rotation stuck in my old senile mind.

I see that things are even worse than what I describe above as I errooneously reference Problem 1 in Problem 2. Actually, as it stands that turned out to not be an erroneous reference, but only because I didn't type the system of ODEs correctly. In any case, things got sort of fouled up. What I'm going to do in this corrected version is (1) add annotations in the problems below to make it possible for you to note how things are getting fouled up and fix the statements of the problems yourself—should you desire to do so. And then work the problems I intended to have for this assignment. If you don't want to do that the annotations should (2) at least make clear how some of the "prompts" are nonsense and should be modified.

I'll also give an alternative version of the assignment in which I actually make corrections instead of just adding annotations. I'll also put some solutions there. I seem to have lost/thrown away my notes for the original composition, so I'm having to compose solutions from scratch. I think I might have had some better/cleverer solutions, but I don't seem to be coming up with them now.

Look for ANNOTATION.

Problem 2 (ANNOTATIONS about corrections) Consider the ODE

$$\gamma' = (-\gamma_1, \gamma_2)$$

for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$. **ANNOTATION:** The basic error is that this should be $\gamma' = (-\gamma_2, \gamma_1)$.

(a) Use mathematical software to **animate** the evolution of the solution of the IVP

$$\begin{cases} \gamma' = L(\gamma), & t \in \mathbb{R} \\ \gamma(0) = (1, i). \end{cases}$$

For each t in some suitable interval $I \subset \mathbb{R}$, plot the projections $\Pi_1 \circ \gamma(t)$ and $\Pi_2 \circ \gamma(t)$ of $\gamma(t)$ in two copies of $\mathbb{C}$ representing the point $\gamma(t)$ in $\mathbb{C}^2$. The **projection operators** $\Pi_j : \mathbb{C}^2 \to \mathbb{C}^2$ for $j = 1, 2$ are given by

$$\Pi_j(z_1, z_2) = z_j \mathbf{e}_j.$$

Use t as an animation parameter for the points $\Pi_j \circ \gamma(t)$ and plot the projection curves $\Gamma_j = \{\Pi_j \circ \gamma(t) : t \in I\}$.

(b) Let Σ denote the collection of all solutions you found in part **(b)** of Problem 1. Show Σ is a complex linear subspace of $C^\infty(\mathbb{R} \to \mathbb{C}^2)$ of dimension 2.

ANNOTATION: This problem makes reasonable sense as stated if the ODEs in Problems 1 and 2 are the same. One possible fix is to also change the ODE in Problem 1. Then I guess parts **(a)** and **(b)** still both make sense, but are somewhat easier than intended. Assuming this problem has a different ODE from Problem 1, then part **(b)** should read: "Let Σ denote the collection of all solutions of $\gamma' = (-\gamma_2, \gamma_1)$. Show Σ is a complex linear subspace of $C^\infty(\mathbb{R} \to \mathbb{C}^2)$ of dimension 2."

Problem 3 Consider the projection curves Γ_1 and Γ_2 for a general solution γ in the solution space Σ as considered in Problem 2 above.

(a) Show Γ_j for $j = 1, 2$ is either an ellipse or a line segment which is a degenerate ellipse.

ANNOTATION: You won't get ellipses with $\gamma' = (-\gamma_1, \gamma_2)$. If you stick with this simpler equation, I guess you get something like "... either a ray or a single point."

(b) Show Γ_1 and Γ_2 are congruent.

ANNOTATION: I guess this is still more or less true for $\gamma' = (-\gamma_1, \gamma_2)$, though maybe you can get a ray for one projection and a single point $0 \in \mathbb{C}$ for the other.

(c) Show the parameterizations $\Pi_1 \circ \gamma$ and $\Pi_2 \circ \gamma$ of the projection ellipses Γ_1 and Γ_2 respectively are 90 degrees out of phase.

ANNOTATION: I guess this doesn't work at all for $\gamma' = (-\gamma_1, \gamma_2)$ or at least there is not much of an analogue. Let's see, if Γ_1 is the ray through

$c_1 = |c_1|e^{i\theta_1} \in \mathbb{C}\backslash\{0\}$ and Γ_2 is a ray through $c_2 = |c_2|e^{i\theta_2} \in \mathbb{C}\backslash\{0\}$, then probably

$$\Gamma_2 = \{e^{i(\theta_2-\theta_1)}z : z \in \Gamma_1\}.$$

Note also that the solution determining Γ_1 is a contracting or decaying straight line solution while the solution determining the ray Γ_2 is an expanding or growing straight line solution. The fact that these two projections are not so coherently correlated is a consequence of the fact that the system $\gamma' = (-\gamma_1, \gamma_2)$ is not coupled.

(d) Characterize the solutions in Σ leading to clockwise parameterizations of the projection ellipses.

ANNOTATION: Again I don't see any particular analogue of this assertion for $\gamma' = (-\gamma_1, \gamma_2)$. It just happens that the solutions have the same speed (or somehow related speeds) because one has eigenvalues $\lambda = \pm 1$. I don't see much interesting to do with this.

Problem 4 Consider the ODE $\gamma' = (-\gamma_1, \gamma_2)$ for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$. **ANNOTATION:** Again this should be $\gamma' = (-\gamma_2, \gamma_1)$.

(a) Show

$$\mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{C}^2 : x_1, x_2 \in \mathbb{R}\} \tag{1}$$

is an **invariant subset** for $L : \mathbb{C}^2 \to \mathbb{C}^2$ by $L(z_1, z_2) = (-z_1, z_2)$.

ANNOTATION: This makes sense as stated. For the alternative ODE $\gamma' = (-\gamma_2, \gamma_1)$ one should use $L(z_1, z_2) = (-z_2, z_1)$.

(b) Is $\mathbb{R}^2$ given in (1) a subspace of $\mathbb{C}^2$?

ANNOTATION: This is a general question which stands along independent of any errors or details in the composition of the assignment.

(c) Find all solutions $\gamma \in \Sigma$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ where $\mathbb{R}^2$ is given in (1) and Σ is the solution space from part **(b)** of Problem 2.

ANNOTATION: I think this statement is okay for both ODEs, but it is much more difficult for $\gamma' = (-\gamma_2, \gamma_1)$.

Problem 5 Consider the ODE $\gamma' = (-\gamma_1, \gamma_2)$ for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$. **ANNOTATION:** Again this should be $\gamma' = (-\gamma_2, \gamma_1)$.

(a) Let P denote the collection of all solutions you found in part **(c)** of Problem 4. Show P is a real linear subspace of $C^\infty(\mathbb{R} \to \mathbb{R}^2)$ of dimension 2.

(b) Is P a subspace of Σ?

ANNOTATION: I think this problem basically makes sense for either ODE, and the solution is about the same too.

Problem 6 Consider the ODE

$$\gamma' = (-i\gamma_1, i\gamma_2) \tag{2}$$

for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$.

(a) Show $B = \{\phi_1, \phi_2\} \subset C^\infty(\mathbb{R} \to \mathbb{C})$ with

$$\phi_1(t) = (\cos t - i \sin t, 0) \qquad \text{and} \qquad \phi_2(t) = (0, \cos t + i \sin t)$$

is a basis of solutions for (2).

(b) Consider the linear isometry $K : \mathbb{C}^2 \to \mathbb{C}^2$ determined by

$$K(\mathbf{e}_1) = \frac{1}{\sqrt{2}}(1, i) \qquad \text{and} \qquad K(\mathbf{e}_2) = \frac{1}{\sqrt{2}}(1, -i).$$

Show $\Phi : \Sigma \to \text{span}\{\phi_1, \phi_2\}$ by $\Phi(\gamma) = K^{-1} \circ \gamma$ is a linear isomorphism.

(c) Find the invariant subset

$$K^{-1}(\mathbb{R}^2) = \{K^{-1}(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

corresponding to $\mathbb{R}^2$ in (1) and verify $K^{-1}(\mathbb{R}^2)$ is invariant for an appropriate linear function $M : \mathbb{C}^2 \to \mathbb{C}^2$.

ANNOTATION: I believe this problem makes sense as stated, though it may seem anomolous that the inclusion (1) is from Problem 4 which may seem unrelated to this problem. If the correct(ed) ODE $\gamma' = (-\gamma_2, \gamma_1)$ is used, then the vectors appearing in part **(b)** should be familiar as eigenvectors of the linear operator $L : \mathbb{C}^2 \to \mathbb{C}^2$ with $L(z_1, z_2) = (-z_2, z_1)$ as appearing in the correct(ed) version of Problem 4.

Definition 2 A **higher order linear ordinary differential equation** is an ODE of the form

$$\sum_{j=0}^{k} a_j(t) \frac{d^j u}{dt^j} = f(t) \tag{3}$$

typically for a function $u \in C^k(I \to \mathbb{C})$ for some interval $I \subset \mathbb{R}$ and **coefficient functions** $a_j, f \in C^0(I \to \mathbb{C})$ for $j = 0, 1, 2, \ldots, k$.

A point $t_0 \in I$ is said to be a **regular** point for the ODE (3) if $a_k(t_0) \neq 0$.

The ODE is said to be **homogeneous** if $f \equiv 0$. Given the ODE (3) the **associated homogeneous equation** is

$$\sum_{j=0}^{k} a_j(t) \frac{d^j u}{dt^j} = 0.$$

Problem 7 Is the notion of a linear system of ODEs from Assignment 6 consistent with/equivalent to the definition of the linear ODE above in the overlapping case of a single equation of order 1?

Problem 8 Show the k-th order linear ODE (3) above on a regular open interval $I \subset \mathbb{R}$ on which $a_k(t) \neq 0$ is equivalent to a first order linear system in the following sense(s):

(a) Given a solution $u \in C^k(I \to \mathbb{C})$ for the k-th order ODE (3), the function $\mathbf{x} \in C^1(I \to \mathbb{C}^k)$ with

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_k(t)) = (u(t), u'(t), u''(t), \ldots, u^{(k-1)}(t)) \tag{4}$$

satisfies a first order linear system of ODEs. (Ideintify the system by giving the coefficient matrix $a_{j\ell} = a_{j\ell}(t)$ and the inhomogeneity F.)

(b) Given a solution $\gamma \in C^1(I \to \mathbb{C}^k)$ of the first order linear system you found in part **(a)**, the first component function $u = \gamma_1 \in C^1(I \to \mathbb{C})$ satisfies $\gamma_1 \in C^k(I \to \mathbb{C})$ and equation (3).

(c) If Σ denotes the collection of solutions to the associated homogeneous equation

$$\sum_{j=0}^{k} a_j(t) \frac{d^j u}{dt^j} = 0.$$

(i) $\Phi : \Sigma \to \Gamma$ by $\Phi(u) = \gamma$ with γ having values given in (4) is a linear isomorphism onto the solution set for the associated homogeneous first order system.

(ii) Given any $t_0 \in \mathbb{C}$, the function $\Psi : \Sigma \to \mathbb{C}^n$ by

$$\Psi(u) = (u(t_0), u'(t_0), u''(t_0), \ldots, u^{(k-1)}(t_0))$$

is a linear isomorphism.

(You'll need your existence and uniqueness theorem from part **(a)** of Problem 10 on Assignment 6 for this.)

The IVP for (3) has the form

$$\begin{cases} \displaystyle\sum_{j=0}^{k} a_j \frac{d^j u}{dt^j} = f, & t \in I \\ \\ u(t_0) = z_1, u'(t_0) = z_2, u''(t_0) = z_3, \ldots, u^{(k-1)}(t_0) = z_k. \end{cases} \tag{5}$$

Problem 9 Assume u_p is a particular solution of the higher order linear ODE (3).

(a) Show every solution u of (3) has the form $u = u_h + u_p$ where u_h is a solution of the associated homogeneous ODE. Hint: Assume u is any solution and write $u_h = u - u_p$.

(b) If u satisfies the IVP (5) and $u = u_h + u_p$, find an IVP satisfied by u_h.

Problem 10 In the special case of a higher order linear ODE with coefficient functions $a_0, a_1, \ldots, a_k$ constant (and $a_k \neq 0$) and $f \in C^\infty(I \to \mathbb{C})$ with $I = [0, \infty)$ and $t_0 = 0$ there is a linear function

$$\mathcal{T} : \Sigma \to W^{\text{dual}}$$

by

$$\mathcal{T}[u] = \phi$$

where

$$\phi[v] = \int_0^\infty v(t)\overline{u(t)}\, dt, \tag{6}$$

W is the subspace of $C^\infty(I \to \mathbb{C})$ containing functions decaying fast enough to make the integral(s) in (6) converge, and Σ is the collection of solutions of the IVP (5).

(a) Assuming $\mathcal{T}$ extends to a larger domain containing

$$u' = \frac{du}{dt}$$

with the value(s) of $\mathcal{T}[u']$ given by the same integral formula (6) with u' in place of u, show $\mathcal{T}[u'] = \phi'$ with

$$\phi'[e^{-st}] = -\overline{u(0)} + s\phi[e^{-st}].$$

Note that we are abusing notation slightly here by letting e^{-st} denote the (rapidly decaying test) function $v : [0, \infty) \to \mathbb{C}$ with $v(t) = e^{-st}$.

(b) Similarly, assuming $f : [0, \infty) \to \mathbb{C}$ is in the domain of $\mathcal{T}$ with the values of $\mathcal{T}[f]$ given by the same integral formula (6) with f in place of u, the function $g : [0, \infty) \to \mathbb{C}$ with values

$$g(t) = H(t - t_1)f(t - t_1)$$

where $H : \mathbb{R} \to \mathbb{R}$ and $t_1 > 0$ by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

should also have $\mathcal{T}[g]$ well-defined by an integral. Under these assumptions show $\mathcal{T}[g] = G$ with

$$G[e^{-st}] = e^{-st_1}\mathcal{T}[f][e^{-st}].$$

This construction leaves open the possibility of deriving a necessary condition in W^{dual} imposed by the existence of a solution of the IVP for a constant coefficient linear ODE. In the subspace W of rapidly decaying functions in $C^\infty(I \to \mathbb{C})$ there is an inner product given by

$$\langle v, w \rangle = \int_0^\infty v(t)\overline{w(t)}\, dt$$

and many functionals in W^{dual} can be built using this inner product. One does not have Riesz representation, and there are other functionals in W^{dual} not given by integral representation. Most notably one finds the evaluation functionals $\mathcal{E}_{t_1} : W \to \mathbb{C}$ with $\mathcal{E}[v] = v(t_1)$.

(c) Derive the necessary condition

$$U[e^{-st}] = \frac{1}{s - i}$$

for $U = \mathcal{T}[u]$ when u is the solution of

$$\begin{cases} u'' + u = 0, & t \in [0, \infty) \\ u(0) = 1,\ u'(0) = -i. \end{cases}$$