Assignment 7 alt.: Linear theory of ODEs and higher order equations
Alternative/Corrected Version of Problems 1-5
Due Thursday November 12, 2026

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Definition 1 The **standard inner product** $\langle \cdot\,,\,\cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ on $\mathbb{C}^n$ is given by

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

where $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$.

Two vectors $z, w \in \mathbb{C}^n$ are said to be **orthogonal** if $\langle z, w \rangle = 0$.

A set of vectors $\{v_1, v_2, \ldots, v_k\} \subset \mathbb{C}^n$ is said to be **orthonormal** if

$$\langle v_j, v_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

Problem 1 Consider $L : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$L(z_1, z_2) = (-z_1, z_2).$$

(a) Find an orthonormal basis for $\mathbb{C}^2$ consisting of eigenvectors for L.

(b) Find all solutions $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$ of the linear system $\gamma' = L(\gamma)$.

Problem 2 Consider the ODE

$$\gamma' = L(\gamma)$$

for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$ with $L \in \mathcal{L}(\mathbb{C}^2)$ given by

$$L(x_1, x_2) = (-x_2, x_1).$$

(a) Use mathematical software to **animate** the evolution of the solution of the IVP

$$\begin{cases} \gamma' = L(\gamma), & t \in \mathbb{R} \\ \gamma(0) = (1, i). \end{cases}$$

For each t in some suitable interval $I \subset \mathbb{R}$, plot the projections $\Pi_1 \circ \gamma(t)$ and $\Pi_2 \circ \gamma(t)$ of $\gamma(t)$ in two copies of $\mathbb{C}$ representing the point $\gamma(t)$ in $\mathbb{C}^2$. The **projection operators** $\Pi_j : \mathbb{C}^2 \to \mathbb{C}^2$ for $j = 1, 2$ are given by

$$\Pi_j(z_1, z_2) = z_j \mathbf{e}_j.$$

Use t as an animation parameter for the points $\Pi_j \circ \gamma(t)$ and plot the projection curves $\Gamma_j = \{\Pi_j \circ \gamma(t) : t \in I\}$.

(b) Find an orthonormal basis for $\mathbb{C}^2$ consisting of eigenvectors for L.

(c) Let Σ denote the collection of all solutions. Show Σ is a complex linear subspace of $C^\infty(\mathbb{R} \to \mathbb{C}^2)$ of dimension 2.

Solution: This problem was intended to follow up on (or build on) Problems 9 and 10 of Assignment 6. The solution of Problem 10 of Assignment 6 applied to the complex ODE $\gamma' = (-\gamma_2, \gamma_1)$ should look something like this: Let $\gamma_1 = \xi_1 + i\eta_1$ and $\gamma_2 = \xi_2 + i\eta_2$. Then one can write the ODE as

$$\begin{cases} \xi_1' + i\eta_1' &= -\xi_2 - i\eta_2 \\ \xi_2' + i\eta_2' &= \xi_1 + i\eta_1. \end{cases}$$

Equating real and imaginary parts this is equivalent to the real system of ODEs

$$\begin{cases} \xi_1' &= -\xi_2 \\ \eta_1' &= -\eta_2 \\ \xi_2' &= \xi_1 \\ \eta_2' &= \eta_1 \end{cases}$$

for $\mathbf{y} : \mathbb{R} \to \mathbb{R}^4$ with $by(t) = (\xi_1(t), \eta_1(t), \xi_2(t), \eta_2(t))$. Incorporating the initial values

$$\xi_1(0) = 1, \quad \eta_1(0) = 0, \quad \xi_2(0) = 0, \quad \eta_2(0) = 1$$

the resulting initial value problem can be solved numerically, and using the identification of $\mathbb{C}$ with $\mathbb{R}^2$ given by $x + iy \mapsto (x, y)$ one can plot the projections.

(a) See the Mathematica notebook `ass2aProb2a-26.nb`.

(b) Say $\lambda \in \mathbb{C}$ is an eigenvalue and $v = (v_1, v_2) \in \mathbb{C}^2$ is an eigenvector. Then we need

$$\begin{cases} -v_2 &= \lambda v_1 \\ v_1 &= \lambda v_2. \end{cases}$$

This means we need $-v_2 = \lambda^2 v_2$. Assuming $v_2 \neq 0 \in \mathbb{C}$, this means we need $\lambda = \pm i$. In fact, taking $\lambda = -i$, we find (taking $v_1 = 1$)

$$v = (1, i)$$

is a corresponding eigenvector. Thus, $(-i, (1, i))$ is an eigenvalue/eigenvector pair. Similarly, $(i, (1, -i))$ is an eigenvalue/eigenvector pair. It also happens to be the case that in $\mathbb{C}^2$

$$\langle (1, i), (1, -i) \rangle = (1)(1) + (i)\overline{(-i)} = 1 - 1 = 0.$$

Thus, these two vectors are orthogonal in $\mathbb{C}^2$, and an orthonormal basis is given by

$$\left\{ \frac{1}{\sqrt{2}}(1, i), \frac{1}{\sqrt{2}}(1, -i) \right\}.$$

This is of course an orthonormal basis consisting of eigenvectors.

(c) Consider

$$\left\{ \frac{c_1}{\sqrt{2}} e^{-it}(1, i) + \frac{c_2}{\sqrt{2}} e^{it}(1, -i) : c_1, c_2 \in \mathbb{C} \right\}.$$

I claim this is the solution set Σ. Naturally, I'm abusing notation a little bit here. What I really mean is the collection of functions $\gamma : \mathbb{R} \to \mathbb{C}^2$ with values given by

$$\gamma(t) = \frac{c_1}{\sqrt{2}} e^{-it}(1, i) + \frac{c_2}{\sqrt{2}} e^{it}(1, -i)$$

for some fixed $c_1, c_2 \in \mathbb{C}$. Hopefully my meaning is clear.

First of all, differentiating using the chain rule of Problem 9 of Assignment 6 we get

$$\begin{aligned}\frac{d}{dt}\left[\frac{c_1}{\sqrt{2}}e^{-it}(1,i)+\frac{c_2}{\sqrt{2}}e^{it}(1,-i)\right] &= \frac{c_1}{\sqrt{2}}(-i)e^{-it}(1,i)+\frac{c_2}{\sqrt{2}}(i)e^{it}(1,-i)\\ &= \frac{c_1}{\sqrt{2}}e^{-it}(-i,1)+\frac{c_2}{\sqrt{2}}e^{it}(i,1)\\ &= \frac{c_1}{\sqrt{2}}e^{-it}L(1,i)+\frac{c_2}{\sqrt{2}}e^{it}L(1,i)\\ &= L\left(\frac{c_1}{\sqrt{2}}e^{-it}(1,i)+\frac{c_2}{\sqrt{2}}e^{it}(1,-i)\right).\end{aligned}$$

This means every function $\gamma : \mathbb{R} \to \mathbb{C}^2$ with values given by

$$\gamma(t) = \frac{c_1}{\sqrt{2}}e^{-it}(1,i)+\frac{c_2}{\sqrt{2}}e^{it}(1,-i)$$

is in Σ. This solution, incidentally has initial value

$$\gamma(0) = \frac{c_1}{\sqrt{2}}(1,i)+\frac{c_2}{\sqrt{2}}(1,-i) = \frac{1}{\sqrt{2}}\left(c_1+c_2,(c_1-c_2)i\right)$$

or more generally

$$\gamma(t_0) = \frac{c_1}{\sqrt{2}}e^{-it_0}(1,i)+\frac{c_2}{\sqrt{2}}e^{it_0}(1,-i) = \frac{1}{\sqrt{2}}\left(c_1e^{-it_0}+c_2e^{it_0},\left(c_1e^{-it_0}-c_2e^{it_0}\right)i\right).$$

Now any $\gamma \in \Sigma$ satisfies some initial value problem

$$\begin{cases}\gamma' = L\circ\gamma, & t\in\mathbb{R}\\ \gamma(t_0) = (w_1,w_2)\end{cases}$$

where $(w_1, w_2) \in \mathbb{C}^2$. In fact, the linear existence and uniqueness theorem tells us that every such solution is defined for all $t \in \mathbb{R}$. I'm not sure I've stated this theorem, but the conclusion is that a linear system has a unique solution on any interval where the coefficients are continuous, and this system is linear homogeneous and the coefficients are constant. In any case, we can just solve the problem: Take

$$c_1 = \frac{e^{it_0}}{\sqrt{2}}(w_1-iw_2) \qquad \text{and} \qquad c_2 = \frac{e^{-it_0}}{\sqrt{2}}(w_1+iw_2).$$

Then the function $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$ with

$$\gamma(t) = \frac{c_1}{\sqrt{2}} e^{-it}(1, i) + \frac{c_2}{\sqrt{2}} e^{it}(1, -i)$$

is the unique solution of the initial value problem, and in particular we see

$$\Sigma = \left\{ \frac{c_1}{\sqrt{2}} e^{-it}(1, i) + \frac{c_2}{\sqrt{2}} e^{it}(1, -i) : c_1, c_2 \in \mathbb{C} \right\}$$

as claimed. Incidentally, I got the values of c_1 and c_2 here by taking the form of the solution, i.e., the linear combination of straight line solutions, evaluating at t_0 and then taking inner products in $\mathbb{C}^2$:

$$c_1 e^{-it_0} = \left\langle (w_1, w_2), \frac{1}{\sqrt{2}}(1, i) \right\rangle = \frac{1}{\sqrt{2}}(w_1 - iw_2)$$

and

$$c_2 e^{it_0} = \left\langle (w_1, w_2), \frac{1}{\sqrt{2}}(1, -i) \right\rangle = \frac{1}{\sqrt{2}}(w_1 + iw_2).$$

(It helps to have an orthonormal basis.)

Every solution is a linear combination of "straight line" solutions.

Problem 3 Consider the projection curves Γ_1 and Γ_2 for a general solution γ in the solution space Σ as considered in Problem 2 above.

(a) Show Γ_j for $j = 1, 2$ is either an ellipse or a line segment which is a degenerate ellipse.

(b) Show Γ_1 and Γ_2 are congruent.

(c) Show the parameterizations $\Pi_1 \circ \gamma$ and $\Pi_2 \circ \gamma$ of the projection ellipses Γ_1 and Γ_2 respectively are 90 degrees out of phase.

(d) Characterize the solutions in Σ leading to clockwise parameterizations of the projection ellipses.

Solution: We know every solution has the form

$$\gamma(t) = \frac{c_1}{\sqrt{2}} e^{-it}(1, i) + \frac{c_2}{\sqrt{2}} e^{it}(1, -i)$$

for some $c_1, c_2 \in \mathbb{C}$. We can write $c_1 = |c_1|e^{i\theta_1}$ and $c_2 = |c_2|e^{i\theta_2}$ for some $\theta_1, \theta_2 \in \mathbb{R}$. Then

$$\begin{aligned}
\gamma(t) &= \frac{1}{\sqrt{2}}(|c_1|e^{-i(t-\theta_1)}(1, i) + |c_2|e^{i(t+\theta_2)}(1, -i)) \\
&= \frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left(|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)}(1, i) + |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)}(1, -i)\right) \\
&= \left(\frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left(|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)} + |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)}\right),\right. \\
&\qquad \left. \frac{i}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left(|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)} - |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)}\right)\right).
\end{aligned}$$

In particular,

$$\Pi_1 \circ \gamma(t) = \frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left(|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)} + |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)}\right)$$

and

$$\Pi_2 \circ \gamma(t) = \frac{i}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left(|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)} - |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)}\right)$$

Notice that the points in $\Gamma_1 = \{\Pi_1 \circ \gamma(t) : t \in \mathbb{R}\}$ are a scaling by $1/\sqrt{2}$ and a (counterclockwise) rotation by a fixed angle $(\theta_1 + \theta_2)/2$ of the points in

$$\left\{|c_1|e^{-i\left(t+\frac{\theta_2-\theta_1}{2}\right)} + |c_2|e^{i\left(t+\frac{\theta_2-\theta_1}{2}\right)} : t \in \mathbb{R}\right\}.$$

This set can also be written as

$$\left\{|c_1|e^{-i\tau} + |c_2|e^{i\tau} : \tau \in \mathbb{R}\right\}$$

where of course we have used the substitution

$$\tau = t + \frac{\theta_2 - \theta_1}{2}.$$

That is, we are interested in the points

$$(|c_1| + |c_2|)\cos\tau + i(|c_2| - |c_1|)\sin\tau.$$

(a) If $|c_1| = |c_2|$, then the points

$$(|c_1| + |c_2|)\cos\tau + i(|c_2| - |c_1|)\sin\tau, \qquad \tau \in \mathbb{R}$$

describe a segment

$$\{a + bi \in \mathbb{C} : b = 0 \text{ and } -2|c_1| \le a \le 2|c_1|\}$$

along the real axis (which is only the origin in case $|c_1| = |c_2| = 0$).

If $|c_2| > |c_1|$, then setting $x_1 = (|c_1| + |c_2|)\cos\tau$ and $x_2 = (|c_2| - |c_1|)\sin\tau$ we see

$$\frac{x_1^2}{(|c_1| + |c_2|)^2} + \frac{x_2^2}{(|c_2| - |c_1|)^2} = 1,$$

so $\mathbf{x} : \mathbb{R} \to \mathbb{C}$ by

$$\mathbf{x}(\tau) = |c_1|e^{-i\tau} + |c_2|e^{i\tau}$$

gives a counterclockwise parameterization of an ellipse with major axis the segment

$$\{a + bi \in \mathbb{C} : b = 0 \text{ and } -(|c_1| + |c_2|) \le a \le |c_1| + |c_2|\}$$

along the real axis and minor axis the segment

$$\{a + bi \in \mathbb{C} : a = 0 \text{ and } -(|c_2| - |c_1|) \le b \le |c_2| - |c_1|\}$$

along the imaginary axis. In particular Γ_1 is a scaling and rotation of this ellipse, so Γ_1 is an ellipse.

If $|c_2| < |c_1|$, then the ellipse is essentially the same, but the minor axis should be written as

$$\{a + bi \in \mathbb{C} : a = 0 \text{ and } -(|c_1| - |c_2|) \le b \le |c_1| - |c_2|\}$$

and the parameterization is given in a clockwise direction.

Turning to Γ_2, we are interested in the points

$$\mathbf{y}(\tau) = (|c_1| - |c_2|)\cos\tau - i(|c_1| + |c_2|)\sin\tau.$$

This is a parameterization of an ellipse with $\mathbf{y}(0) = |c_1| - |c_2|$ on the minor axis

$$\{a + bi \in \mathbb{C} : b = 0 \text{ and } -|\,|c_1| - |c_2|\,| \leq b \leq |\,|c_1| - |c_2|\,|\}$$

and vertical major axis

$$\{a + bi \in \mathbb{C} : a = 0 \text{ and } -(|c_1| + |c_2|) \leq b \leq |c_1| + |c_2|\}.$$

Again, if $|c_2| > |c_1|$, the parameterization is in a counterclockwise direction, and if $|c_2| < |c_1|$, then the parameterization is in a clockwise direction. In the case $|c_1| = |c_2|$ the ellipse degenerates into a vertical line segment

$$\{a + bi \in \mathbb{C} : a = 0 \text{ and } -2|c_1| \leq b \leq 2|c_1|\}.$$

(b) From the description above the unscaled ellipse

$$A_2 = \{\mathbf{y}(\tau) : \tau \in \mathbb{R}\}$$

is a counterclockwise rotation by $\pi/2$ of the unscaled ellipse

$$A_1 = \{\mathbf{x}(\tau) : \tau \in \mathbb{R}\}.$$

The projection Γ_1 is essentially a scaling (and rotation) of A_1. Recall that the scaling factor is

$$\frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}}$$

with the factor $1/\sqrt{2}$ giving a dilation and the unit length complex number

$$e^{i\frac{\theta_1+\theta_2}{2}} = \cos\frac{\theta_1 + \theta_2}{2} + i\sin\frac{\theta_1 + \theta_2}{2}$$

giving a counterclockwise rotation by the fixed angle $(\theta_1 + \theta_2)/2$. Of course, if $\theta_1 + \theta_2 < 0$, the rotation is effectively a clockwise rotation, but that does not really make a difference.

The scaling factor relating A_2 to Γ_2 is

$$\frac{i}{\sqrt{2}}e^{i\frac{\theta_1+\theta_2}{2}} = \frac{1}{\sqrt{2}}e^{i\frac{\theta_1+\theta_2+\pi}{2}}.$$

The additional factor of $i = e^{i\pi/2}$ corresponds to a counterclockwise rotation by $\pi/2$, so that Γ_1 and Γ_2 are essentially the same ellipse in the complex plane and are thus congruent.

Technically, the ellipses Γ_1 and Γ_2 are not the same ellipse because they are in two different projection planes. In particular Γ_1 is in the complex projection plane

$$\{(z,0) : z \in \mathbb{C}\} \subset \mathbb{C}^2$$

associated with the projection Π_1, and Γ_2 is in the complex projection plane

$$\{(0,z) : z \in \mathbb{C}\} \subset \mathbb{C}^2$$

associated with Π_2.

(c) Finally, considering the natural identification of Γ_1 with Γ_2 by

$$\Gamma_1 \ni (z,0) \mapsto (0,z) \in \Gamma_2,$$

we can consider the precise parameterizations in each case. If $|c_2| > |c_1|$, then both parameterizations are counterclockwise as mentioned above. Furthermore, the point

$$\Pi_1 \circ \gamma\left(\frac{\theta_1-\theta_2}{2}\right) = (z_1, 0)$$

arises as the scaling of $\mathbf{x}(0) = |c_1| + |c_2| > 0$. Specifically,

$$z_1 = \frac{|c_1|+|c_2|}{\sqrt{2}}e^{i\frac{\theta_1+\theta_2}{2}}.$$

The point

$$\Pi_2 \circ \gamma\left(\frac{\theta_1-\theta_2}{2}\right) = (0, z_2)$$

on the other hand, arises as the scaling of $\mathbf{y}(0) = |c_1| - |c_2| < 0$. Specifically,

$$z_2 = -i\frac{|c_2|-|c_1|}{\sqrt{2}}e^{i\frac{\theta_1+\theta_2}{2}}.$$

Notice the "extra" factor of i corresponding to a rotation by $\pi/2$ in this expression. This means that while z_1 is at one end of the major axis, say at $|c_1| + |c_2|$ if we ignore the scaling, then z_2 is at the end of the minor axis, for example at $-(|c_2| - |c_1|)i$ if we ignore the dilation and rotation by $(\theta_1 + \theta_2)/2$, but we can't ignore the rotation associated with the factor i.

Thus, we would say the parameterization $\Pi_2 \circ \gamma$ is lagging behind the parameterization $\Pi_1 \circ \gamma$, and this relation extends to all arguments t.

Notice this is not the situation for the solution in Problem 2 where $\gamma(0) = (1, i)$. In that case however, taking account of the calculation in Problem 2 part **(c)**, we have $(w_1, w_2) = (1, i)$ and

$$c_1 = \frac{1}{\sqrt{2}}(1 - i^2) = \sqrt{2} \qquad \text{and} \qquad c_2 = \frac{1}{\sqrt{2}}(1 + i^2) = 0.$$

It is not the case that $|c_2| > |c_1|$, and in fact the circles are parameterized clockwsise instead of counterclockwise.

In general if $|c_2| < |c_1|$, then the projection ellipses will be parameterized in the c ounterclockwise direction, and the relation of the points starting at $t = (\theta_1 - \theta_2)/2$ is illustrated by $z_1 = (|c_1| + |c_2|)/\sqrt{2}$ (in the case $\theta_1 + \theta_2 = 0$) and $z_2 = i(|c_1| - |c_2|)/\sqrt{2}$. Thus, the second factor again lags behind by $\pi/2$ in the argument t.

Taking $c_1 = 1$ and $c_2 = 2$ where in fact $\theta_1 = \theta_2 = 0$, we can animate the solution to get a ellipse with nonunitary eccentricity and a counterclockwise parameteriztion. I've included this in the Mathematica notebook `ass2aProb2a-26.nb`.

Postscript: First of all, here is a better explanation/solution for part **(c)**: We have Γ_1 parameterized by $\gamma_1 : \mathbb{R} \to \mathbb{C}^2$ with

$$\begin{aligned}\gamma_1(t) &= \Pi_1 \circ \gamma(t) \\ &= \left(\frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left((|c_1| + |c_2|) \cos\left(t + \frac{\theta_2 - \theta_1}{2} \right) \right.\right. \\ &\qquad \left.\left. + i(|c_2| - |c_1|) \sin\left(t + \frac{\theta_2 - \theta_1}{2} \right) \right), 0 \right).\end{aligned}$$

This is obtained by starting with the expression

$$\mathbf{x}(\tau) = (|c_1| + |c_2|) \cos\tau + i(|c_2| - |c_1|) \sin\tau,$$

replacing τ with

$$\tau = t + \frac{\theta_2 - \theta_1}{2},$$

reintroducing the scaling by $e^{i\frac{\theta_1+\theta_2}{2}}/\sqrt{2}$, and finally using the inclusion map from $\mathbb{C}$ into the first projection plane of $\mathbb{C}^2$. This expression might have been obtained from the beginning, but the introduction of τ and stripping off the scaling etc. was used above to make the calculations more transparent.

Following a similar repackaging procedure we obtain a parameterization of Γ_2 by $\gamma_2 : \mathbb{R} \to \mathbb{C}^2$ with

$$\begin{aligned}
\gamma_2(t) &= \Pi_2 \circ \gamma(t) \\
&= \left(0, \frac{i}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left((|c_1| - |c_2|) \cos\left(t + \frac{\theta_2 - \theta_1}{2}\right) \right.\right. \\
&\qquad \left.\left. -i(|c_1| + |c_2|) \sin\left(t + \frac{\theta_2 - \theta_1}{2}\right)\right)\right) \\
&= \left(0, \frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left((|c_1| + |c_2|) \sin\left(t + \frac{\theta_2 - \theta_1}{2}\right) \right.\right. \\
&\qquad \left.\left. +i(|c_1| - |c_2|) \cos\left(t + \frac{\theta_2 - \theta_1}{2}\right)\right)\right).
\end{aligned}$$

In the last exprssion, we have multiplied through the "extra" $\pi/2$ rotation factor of $i = e^{i\pi/2}$.

With this second expression we can compute $\gamma_2(t + \pi/2)$. If we get the second component of $\gamma_2(t + \pi/2)$ matching the first component of $\gamma_1(t)$, then this says exactly that the ellipse parameterized by γ_2 is $\pi/2$ out of phase, and in fact lagging by $\pi/2$, compared to the ellipse parameterized by γ_1. Of course, the ellipse parameterized by γ_2 is in the second projection plane and the ellipse parameterized by γ_1 is in the first projection plane. Consequently these are different ellipses, but they are precisely the same ellipse in their respective copies of $\mathbb{C}$, so they are certainly congruent. Okay, let's make the calculation:

$$\cos\left(t + \frac{\pi}{2} + \frac{\theta_2 - \theta_1}{2}\right) = -\sin\left(t + \frac{\theta_2 - \theta_1}{2}\right)$$

and

$$\sin\left(t + \frac{\pi}{2} + \frac{\theta_2 - \theta_1}{2}\right) = \cos\left(t + \frac{\theta_2 - \theta_1}{2}\right).$$

Therefore,

$$\gamma_2(t+\pi/2) = \left(0, \frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left((|c_1|+|c_2|)\cos\left(t+\frac{\theta_2-\theta_1}{2}\right) \right.\right.$$
$$\left.\left. -i(|c_1|-|c_2|)\sin\left(t+\frac{\theta_2-\theta_1}{2}\right)\right)\right)$$

$$= \left(0, \frac{1}{\sqrt{2}} e^{i\frac{\theta_1+\theta_2}{2}} \left((|c_1|+|c_2|)\cos\left(t+\frac{\theta_2-\theta_1}{2}\right) \right.\right.$$
$$\left.\left. +i(|c_2|-|c_1|)\sin\left(t+\frac{\theta_2-\theta_1}{2}\right)\right)\right).$$

The second component of $\gamma_2(t+\pi/2)$ is exactly the first component of $\gamma_1(t)$, so the assertion about the projection points being $\pi/2$ out of phase is established.

My previous explanation was making this calculation (more or less) by comparing the positions of a point in the first projection plane at the end of a major axis with a point in the second projection plane at the end of a minor axis. Such points are of course $\pi/2$ out of phase, but the lagging character and how this relates to all the other points was, shall we say, not as clear as indicated by the calculation above. It was a bit confusing. On the other hand, what is being explained here is a little bit complicated. I hope I've done a better job with this second try.

I didn't formally mention part **(d)** above.

(d) As mentioned above the parameterizations are clockwise parameterizations precisely when $|c_1| > |c_2|$. Incidentally, one can also say that the projection ellipses are circles precisely when

$$|\,|c_1| - |c_2|\,| = |c_1| + |c_2|.$$

This happens whenever on of c_1 or c_2 is zero (and the other is not zero). Recall one gets a segment when $|c_1| = |c_2|$, so this gives I think a pretty complete characterization.

Problem 4 Consider the ODE $\gamma' = (-\gamma_2, \gamma_1)$ for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$.

(a) Show

$$\mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{C}^2 : x_1, x_2 \in \mathbb{R}\} \tag{1}$$

is an **invariant subset** for $L : \mathbb{C}^2 \to \mathbb{C}^2$ by $L(z_1, z_2) = (-z_2, z_1)$.

(b) Is $\mathbb{R}^2$ given in (1) a subspace of $\mathbb{C}^2$?

(c) Find all solutions $\gamma \in \Sigma$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ where $\mathbb{R}^2$ is given in (1) and Σ is the solution space from part **(b)** of Problem 2.

Problem 5 Consider the ODE $\gamma' = (-\gamma_2, \gamma_1)$ for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$.

(a) Let P denote the collection of all solutions you found in part **(c)** of Problem 4. Show P is a real linear subspace of $C^\infty(\mathbb{R} \to \mathbb{R}^2)$ of dimension 2.

(b) Is P a subspace of Σ?

Problem 6 Consider the ODE

$$\gamma' = (-i\gamma_1, i\gamma_2) \tag{2}$$

for $\gamma \in C^\infty(\mathbb{R} \to \mathbb{C}^2)$.

(a) Show $B = \{\phi_1, \phi_2\} \subset C^\infty(\mathbb{R} \to \mathbb{C})$ with

$$\phi_1(t) = (\cos t - i \sin t, 0) \qquad \text{and} \qquad \phi_2(t) = (0, \cos t + i \sin t)$$

is a basis of solutions for (2).

(b) Consider the linear isometry $K : \mathbb{C}^2 \to \mathbb{C}^2$ determined by

$$K(\mathbf{e}_1) = \frac{1}{\sqrt{2}}(1, i) \qquad \text{and} \qquad K(\mathbf{e}_2) = \frac{1}{\sqrt{2}}(1, -i).$$

Show $\Phi : \Sigma \to \operatorname{span}\{\phi_1, \phi_2\}$ by $\Phi(\gamma) = K^{-1} \circ \gamma$ is a linear isomorphism.

(c) Find the invariant subset

$$K^{-1}(\mathbb{R}^2) = \{K^{-1}(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

corresponding to $\mathbb{R}^2$ in (1) and verify $K^{-1}(\mathbb{R}^2)$ is invariant for an appropriate linear function $M : \mathbb{C}^2 \to \mathbb{C}^2$.

Definition 2 A **higher order linear ordinary differential equation** is an ODE of the form

$$\sum_{j=0}^{k} a_j(t)\frac{d^j u}{dt^j} = f(t) \tag{3}$$

typically for a function $u \in C^k(I \to \mathbb{C})$ for some interval $I \subset \mathbb{R}$ and **coefficient functions** $a_j, f \in C^0(I \to \mathbb{C})$ for $j = 0, 1, 2, \ldots, k$.

A point $t_0 \in I$ is said to be a **regular** point for the ODE (3) if $a_k(t_0) \neq 0$.

The ODE is said to be **homogeneous** if $f \equiv 0$. Given the ODE (3) the **associated homogeneous equation** is

$$\sum_{j=0}^{k} a_j(t)\frac{d^j u}{dt^j} = 0.$$

Problem 7 Is the notion of a linear system of ODEs from Assignment 6 consistent with/equivalent to the definition of the linear ODE above in the overlapping case of a single equation of order 1?

Problem 8 Show the k-th order linear ODE (3) above on a regular open interval $I \subset \mathbb{R}$ on which $a_k(t) \neq 0$ is equivalent to a first order linear system in the following sense(s):

(a) Given a solution $u \in C^k(I \to \mathbb{C})$ for the k-th order ODE (3), the function $\mathbf{x} \in C^1(I \to \mathbb{C}^k)$ with

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_k(t)) = (u(t), u'(t), u''(t), \ldots, u^{(k-1)}(t)) \tag{4}$$

satisfies a first order linear system of ODEs. (Ideintify the system by giving the coefficient matrix $a_{j\ell} = a_{j\ell}(t)$ and the inhomogeneity F.)

(b) Given a solution $\gamma \in C^1(I \to \mathbb{C}^k)$ of the first order linear system you found in part **(a)**, the first component function $u = \gamma_1 \in C^1(I \to \mathbb{C})$ satisfies $\gamma_1 \in C^k(I \to \mathbb{C})$ and equation (3).

(c) If Σ denotes the collection of solutions to the associated homogeneous equation

$$\sum_{j=0}^{k} a_j(t) \frac{d^j u}{dt^j} = 0.$$

(i) $\Phi : \Sigma \to \Gamma$ by $\Phi(u) = \gamma$ with γ having values given in (4) is a linear isomorphism onto the solution set for the associated homogeneous first order system.

(ii) Given any $t_0 \in \mathbb{C}$, the function $\Psi : \Sigma \to \mathbb{C}^n$ by

$$\Psi(u) = (u(t_0), u'(t_0), u''(t_0), \ldots, u^{(k-1)}(t_0))$$

is a linear isomorphism.

(You'll need your existence and uniqueness theorem from part **(a)** of Problem 10 on Assignment 6 for this.)

The IVP for (3) has the form

$$\begin{cases} \displaystyle\sum_{j=0}^{k} a_j \frac{d^j u}{dt^j} = f, & t \in I \\ \\ u(t_0) = z_1, u'(t_0) = z_2, u''(t_0) = z_3, \ldots, u^{(k-1)}(t_0) = z_k. \end{cases} \tag{5}$$

Problem 9 Assume u_p is a particular solution of the higher order linear ODE (3).

(a) Show every solution u of (3) has the form $u = u_h + u_p$ where u_h is a solution of the associated homogeneous ODE. Hint: Assume u is any solution and write $u_h = u - u_p$.

(b) If u satisfies the IVP (5) and $u = u_h + u_p$, find an IVP satisfied by u_h.

Problem 10 In the special case of a higher order linear ODE with coefficient functions $a_0, a_1, \ldots, a_k$ constant (and $a_k \neq 0$) and $f \in C^\infty(I \to \mathbb{C})$ with $I = [0, \infty)$ and $t_0 = 0$ there is a linear function

$$\mathcal{T} : \Sigma \to W^{\text{dual}}$$

by

$$\mathcal{T}[u] = \phi$$

where

$$\phi[v] = \int_0^\infty v(t)\overline{u(t)}\, dt, \tag{6}$$

W is the subspace of $C^\infty(I \to \mathbb{C})$ containing functions decaying fast enough to make the integral(s) in (6) converge, and Σ is the collection of solutions of the IVP (5).

(a) Assuming $\mathcal{T}$ extends to a larger domain containing

$$u' = \frac{du}{dt}$$

with the value(s) of $\mathcal{T}[u']$ given by the same integral formula (6) with u' in place of u, show $\mathcal{T}[u'] = \phi'$ with

$$\phi'[e^{-st}] = -\overline{u(0)} + s\phi[e^{-st}].$$

Note that we are abusing notation slightly here by letting e^{-st} denote the (rapidly decaying test) function $v : [0, \infty) \to \mathbb{C}$ with $v(t) = e^{-st}$.

(b) Similarly, assuming $f : [0, \infty) \to \mathbb{C}$ is in the domain of $\mathcal{T}$ with the values of $\mathcal{T}[f]$ given by the same integral formula (6) with f in place of u, the function $g : [0, \infty) \to \mathbb{C}$ with values

$$g(t) = H(t - t_1)f(t - t_1)$$

where $H : \mathbb{R} \to \mathbb{R}$ and $t_1 > 0$ by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

should also have $\mathcal{T}[g]$ well-defined by an integral. Under these assumptions show $\mathcal{T}[g] = G$ with

$$G[e^{-st}] = e^{-st_1}\mathcal{T}[f][e^{-st}].$$

This construction leaves open the possibility of deriving a necessary condition in W^{dual} imposed by the existence of a solution of the IVP for a constant coefficient linear ODE. In the subspace W of rapidly decaying functions in $C^\infty(I \to \mathbb{C})$ there is an inner product given by

$$\langle v, w \rangle = \int_0^\infty v(t)\overline{w(t)}\, dt$$

and many functionals in W^{dual} can be built using this inner product. One does not have Riesz representation, and there are other functionals in W^{dual} not given by integral representation. Most notably one finds the evaluation functionals $\mathcal{E}_{t_1} : W \to \mathbb{C}$ with $\mathcal{E}[v] = v(t_1)$.

(c) Derive the necessary condition

$$U[e^{-st}] = \frac{1}{s - i}$$

for $U = \mathcal{T}[u]$ when u is the solution of

$$\begin{cases} u'' + u = 0, & t \in [0, \infty) \\ u(0) = 1,\ u'(0) = -i. \end{cases}$$