

Assignment 7: Linear theory of ODEs
and higher order equations
Due Thursday November 12, 2026

John McCuan

Definition 1 The **standard inner product** $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ on $\mathbb{C}^n$ is given by

$$\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$$

where $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$.

Two vectors $z, w \in \mathbb{C}^n$ are said to be **orthogonal** if $\langle z, w \rangle = 0$.

A set of vectors $\{v_1, v_2, \dots, v_k\} \subset \mathbb{C}^n$ is said to be **orthonormal** if

$$\langle v_j, v_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

Problem 1 Consider $L : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$L(z_1, z_2) = (-z_1, z_2).$$

- (a) Find an orthonormal basis for $\mathbb{C}^2$ consisting of eigenvectors for L .
- (b) Find all solutions $\gamma \in C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$ of the linear system $\gamma' = L(\gamma)$.

Problem 2 Consider the ODE

$$\gamma' = (-\gamma_1, \gamma_2)$$

for $\gamma \in C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$.

(a) Use mathematical software to **animate** the evolution of the solution of the IVP

$$\begin{cases} \gamma' = L(\gamma), & t \in \mathbb{R} \\ \gamma(0) = (1, i). \end{cases}$$

For each t in some suitable interval $I \subset \mathbb{R}$, plot the projections $\Pi_1 \circ \gamma(t)$ and $\Pi_2 \circ \gamma(t)$ of $\gamma(t)$ in two copies of $\mathbb{C}$ representing the point $\gamma(t)$ in $\mathbb{C}^2$. The **projection operators** $\Pi_j : \mathbb{C}^2 \rightarrow \mathbb{C}$ for $j = 1, 2$ are given by

$$\Pi_j(z_1, z_2) = z_j \mathbf{e}_j.$$

Use t as an animation parameter for the points $\Pi_j \circ \gamma(t)$ and plot the projection curves $\Gamma_j = \{\Pi_j \circ \gamma(t) : t \in I\}$.

(b) Let Σ denote the collection of all solutions you found in part (b) of Problem 1. Show Σ is a complex linear subspace of $C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$ of dimension 2.

Problem 3 Consider the projection curves Γ_1 and Γ_2 for a general solution γ in the solution space Σ as considered in Problem 2 above.

- (a) Show Γ_j for $j = 1, 2$ is either an ellipse or a line segment which is a degenerate ellipse.
- (b) Show Γ_1 and Γ_2 are congruent.
- (c) Show the parameterizations $\Pi_1 \circ \gamma$ and $\Pi_2 \circ \gamma$ of the projection ellipses Γ_1 and Γ_2 respectively are 90 degrees out of phase.
- (d) Characterize the solutions in Σ leading to clockwise parameterizations of the projection ellipses.

Problem 4 Consider the ODE $\gamma' = (-\gamma_1, \gamma_2)$ for $\gamma \in C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$.

(a) Show

$$\mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{C}^2 : x_1, x_2 \in \mathbb{R}\} \quad (1)$$

is an **invariant subset** for $L : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $L(z_1, z_2) = (-z_1, z_2)$.

(b) Is $\mathbb{R}^2$ given in (1) a subspace of $\mathbb{C}^2$?

(c) Find all solutions $\gamma \in \Sigma$ where $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ where $\mathbb{R}^2$ is given in (1) and Σ is the solution space from part (b) of Problem 2.

Problem 5 Consider the ODE $\gamma' = (-\gamma_1, \gamma_2)$ for $\gamma \in C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$.

(a) Let P denote the collection of all solutions you found in part (c) of Problem 4. Show P is a real linear subspace of $C^\infty(\mathbb{R} \rightarrow \mathbb{R}^2)$ of dimension 2.

(b) Is P a subspace of Σ ?

Problem 6 Consider the ODE

$$\gamma' = (-i\gamma_1, i\gamma_2) \quad (2)$$

for $\gamma \in C^\infty(\mathbb{R} \rightarrow \mathbb{C}^2)$.

(a) Show $B = \{\phi_1, \phi_2\} \subset C^\infty(\mathbb{R} \rightarrow \mathbb{C})$ with

$$\phi_1(t) = (\cos t - i \sin t, 0) \quad \text{and} \quad \phi_2(t) = (0, \cos t + i \sin t)$$

is a basis of solutions for (2).

(b) Consider the linear isometry $K : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ determined by

$$K(\mathbf{e}_1) = \frac{1}{\sqrt{2}}(1, i) \quad \text{and} \quad K(\mathbf{e}_2) = \frac{1}{\sqrt{2}}(1, -i).$$

Show $\Phi : \Sigma \rightarrow \text{span}\{\phi_1, \phi_2\}$ by $\Phi(\gamma) = K^{-1} \circ \gamma$ is a linear isomorphism.

(c) Find the invariant subset

$$K^{-1}(\mathbb{R}^2) = \{K^{-1}(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

corresponding to $\mathbb{R}^2$ in (1) and verify $K^{-1}(\mathbb{R}^2)$ is invariant for an appropriate linear function $M : \mathbb{C}^2 \rightarrow \mathbb{C}^2$.

Definition 2 A **higher order linear ordinary differential equation** is an ODE of the form

$$\sum_{j=0}^k a_j(t) \frac{d^j u}{dt^j} = f(t) \quad (3)$$

typically for a function $u \in C^k(I \rightarrow \mathbb{C})$ for some interval $I \subset \mathbb{R}$ and **coefficient functions** $a_j, f \in C^0(I \rightarrow \mathbb{C})$ for $j = 0, 1, 2, \dots, k$.

A point $t_0 \in I$ is said to be a **regular** point for the ODE (3) if $a_k(t_0) \neq 0$.

The ODE is said to be **homogeneous** if $f \equiv 0$. Given the ODE (3) the **associated homogeneous equation** is

$$\sum_{j=0}^k a_j(t) \frac{d^j u}{dt^j} = 0.$$

Problem 7 Is the notion of a linear system of ODEs from Assignment 6 consistent with/equivalent to the definition of the linear ODE above in the overlapping case of a single equation of order 1?

Problem 8 Show the k -th order linear ODE (3) above on a regular open interval $I \subset \mathbb{R}$ on which $a_k(t) \neq 0$ is equivalent to a first order linear system in the following sense(s):

- (a) Given a solution $u \in C^k(I \rightarrow \mathbb{C})$ for the k -th order ODE (3), the function $\mathbf{x} \in C^1(I \rightarrow \mathbb{C}^k)$ with

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_k(t)) = (u(t), u'(t), u''(t), \dots, u^{(k-1)}(t)) \quad (4)$$

satisfies a first order linear system of ODEs. (Identify the system by giving the coefficient matrix $a_{j\ell} = a_{j\ell}(t)$ and the inhomogeneity F .)

- (b) Given a solution $\gamma \in C^1(I \rightarrow \mathbb{C}^k)$ of the first order linear system you found in part (a), the first component function $u = \gamma_1 \in C^1(I \rightarrow \mathbb{C})$ satisfies $\gamma_1 \in C^k(I \rightarrow \mathbb{C})$ and equation (3).
- (c) If Σ denotes the collection of solutions to the associated homogeneous equation

$$\sum_{j=0}^k a_j(t) \frac{d^j u}{dt^j} = 0.$$

- (i) $\Phi : \Sigma \rightarrow \Gamma$ by $\Phi(u) = \gamma$ with γ having values given in (4) is a linear isomorphism onto the solution set for the associated homogeneous first order system.
- (ii) Given any $t_0 \in \mathbb{C}$, the function $\Psi : \Sigma \rightarrow \mathbb{C}^n$ by

$$\Psi(u) = (u(t_0), u'(t_0), u''(t_0), \dots, u^{(k-1)}(t_0))$$

is a linear isomorphism.

(You'll need your existence and uniqueness theorem from part (a) of Problem 10 on Assignment 6 for this.)

The IVP for (3) has the form

$$\begin{cases} \sum_{j=0}^k a_j \frac{d^j u}{dt^j} = f, \\ u(t_0) = z_1, u'(t_0) = z_2, u''(t_0) = z_3, \dots, u^{(k-1)}(t_0) = z_k. \end{cases} \quad t \in I \quad (5)$$

Problem 9 Assume u_p is a particular solution of the higher order linear ODE (3).

(a) Show every solution u of (3) has the form $u = u_h + u_p$ where u_h is a solution of the associated homogeneous ODE. Hint: Assume u is any solution and write $u_h = u - u_p$.

(b) If u satisfies the IVP (5) and $u = u_h + u_p$, find an IVP satisfied by u_h .

Problem 10 In the special case of a higher order linear ODE with coefficient functions $a_0, a_1, \dots, a_k$ constant (and $a_k \neq 0$) and $f \in C^\infty(I \rightarrow \mathbb{C})$ with $I = [0, \infty)$ and $t_0 = 0$ there is a linear function

$$\mathcal{T} : \Sigma \rightarrow W^{\text{dual}}$$

by

$$\mathcal{T}[u] = \phi$$

where

$$\phi[v] = \int_0^\infty v(t) \overline{u(t)} dt, \quad (6)$$

W is the subspace of $C^\infty(I \rightarrow \mathbb{C})$ containing functions decaying fast enough to make the integral(s) in (6) converge, and Σ is the collection of solutions of the IVP (5).

(a) Assuming $\mathcal{T}$ extends to a larger domain containing

$$u' = \frac{du}{dt}$$

with the value(s) of $\mathcal{T}[u']$ given by the same integral formula (6) with u' in place of u , show $\mathcal{T}[u'] = \phi'$ with

$$\phi'[e^{-st}] = -u(0) + \phi[e^{-st}].$$

Note that we are abusing notation slightly here by letting e^{-st} denote the (rapidly decaying test) function $v : [0, \infty) \rightarrow \mathbb{C}$ with $v(t) = e^{-st}$.

- (b) Similarly, assuming $f : [0, \infty) \rightarrow \mathbb{C}$ is in the domain of $\mathcal{T}$ with the values of $\mathcal{T}[f]$ given by the same integral formula (6) with f in place of u , the function $g : [0, \infty) \rightarrow \mathbb{C}$ with values

$$g(t) = H(t - t_1)f(t - t_1)$$

where $H : \mathbb{R} \rightarrow \mathbb{R}$ and $t_1 > 0$ by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

should also have $\mathcal{T}[g]$ well-defined by an integral. Under these assumptions show $\mathcal{T}[g] = G$ with

$$G[e^{-st}] = e^{-st_1} \mathcal{T}[f][e^{-st}].$$

This construction leaves open the possibility of deriving a necessary condition in W^{dual} imposed by the existence of a solution of the IVP for a constant coefficient linear ODE. In the subspace W of rapidly decaying functions in $C^\infty(I \rightarrow \mathbb{C})$ there is an inner product given by

$$\langle v, w \rangle = \int_0^\infty v(t) \overline{w(t)} dt$$

and many functionals in W^{dual} can be built using this inner product. One does not have Riesz representation, and there are other functionals in W^{dual} not given by integral representation. Most notably one finds the evaluation functionals $\mathcal{E}_{t_1} : W \rightarrow \mathbb{C}$ with $\mathcal{E}[v] = v(t_1)$.

- (c) Derive the necessary condition

$$U[e^{-st}] = \frac{s - i}{s^2}$$

for $U = \mathcal{T}[u]$ when u is the solution of

$$\begin{cases} u'' + u = 0, & t \in [0, \infty) \\ u(0) = 1, u'(0) = -i. \end{cases}$$