

Assignment 6: Systems of ODEs

Due Thursday October 29, 2026

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Definition 1 Given $n \in \mathbb{N}$, and $U \subset \mathbb{R}^n$, the set U is **open** if for each $\mathbf{x} \in U$ there exists some $r > 0$ so that

$$B_r(\mathbf{x}) = \{\xi \in \mathbb{R}^n : |\xi - \mathbf{x}| < r\} \subset U.$$

Problem 1 (a) Show that given $a, b \in \mathbb{R}$ with $a < b$, the open interval

$$I = \{t \in \mathbb{R} : a < t < b\}$$

is an open set in $\mathbb{R}^1$.

(b) Show that given $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, the open ball

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{x}| < r\}$$

is an open subset of $\mathbb{R}^n$ where

$$|\mathbf{y} - \mathbf{x}| = \sqrt{\sum_{j=1}^n (y_j - x_j)^2}$$

for

$$\mathbf{y} = (y_1, y_2, \dots, y_n) = \sum_{j=1}^n y_j \mathbf{e}_j \quad \text{and} \quad \mathbf{x} = (x_1, x_2, \dots, x_n) = \sum_{j=1}^n x_j \mathbf{e}_j.$$

Theorem 1 (local existence and uniqueness for a system of ODEs involving n real valued functions) Given an open interval $I \subset \mathbb{R}$ and an open set $U \subset \mathbb{R}^n$ with $(\mathbf{y}, t_0) \in U \times I$, if $F : U \times I \rightarrow \mathbb{R}^n$ satisfies

$$F_j \in C^0(U \times I)$$

and

$$\frac{\partial F_j}{\partial x_k} \in C^0(U \times I)$$

where $F = (F_1, F_2, \dots, F_n)$ and

$$\frac{\partial F_j}{\partial x_k} : U \times I \rightarrow \mathbb{R} \quad \text{by} \quad \frac{\partial F_j}{\partial x_k}(\mathbf{x}, t) = \frac{\partial}{\partial x_k} F_j(\mathbf{x}, t)$$

for $j, k = 1, 2, \dots, n$, then there exists some $\delta > 0$ so that the IVP

$$\begin{cases} \mathbf{x}' = F(\mathbf{x}, t), & t \in (t_0 - \delta, t_0 + \delta) \\ \mathbf{x}(t_0) = \mathbf{y} \end{cases} \quad (1)$$

has a unique solution $\mathbf{x} \in C^1((t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{R}^n)$.

The function F is called the **structure function** for the system of ODEs $\mathbf{x}' = F(\mathbf{x}, t)$.

The system $\mathbf{x}' = F$ is said to be **autonomous** if $F : U \rightarrow \mathbb{R}^n$, so $\mathbf{x}' = F(\mathbf{x})$ has no explicit t dependence on the right.

A system of ODEs of the form $\mathbf{x}' = F(\mathbf{x}, t)$ is said to be **regular**. A more general system of ODEs might have the form $G(\mathbf{x}', \mathbf{x}, t) = \mathbf{0}$ where $G : V \times U \times I \rightarrow \mathbb{R}^n$.

Problem 2 Consider the system of ODEs

$$\begin{cases} x_1' &= x_1^2 - x_2^2 \\ x_2' &= 2x_1x_2. \end{cases} \quad (2)$$

for a vector valued function $\mathbf{x} = (x_1, x_2) \in C^1(\mathbb{R} \rightarrow \mathbb{R}^2)$.

(a) Use mathematical software to draw the image of the solution of the IVP

$$\begin{cases} x_1' = x_1^2 - x_2^2, & t \in (t_0 - 10, t_0 + 10) \\ x_2' = 2x_1x_2, & t \in (t_0 - 10, t_0 + 10) \\ x_1(0) = 0, \quad x_2(0) = 1. \end{cases}$$

(b) What does Theorem 1 tell you about the solution of (2)?

Problem 3 (single complex ODE) Recall the ODE

$$\frac{d\gamma}{dt} = \gamma^2 \quad (3)$$

for $\gamma = \xi + i\eta : \mathbb{R} \rightarrow \mathbb{C}$ from Problem 10 of Assignment 5.

(a) Use mathematical software to draw the circle

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + \left(x_2 - \frac{1}{2}\right)^2 = \frac{1}{4} \right\} \subset \mathbb{R}^2$$

over the image you obtained in part (a) of Problem 2 above. What do you notice?

(b) What is the relation between the system (2) in Problem 2 above and (3)?

Definition 2 Given $\gamma = \xi + i\eta : I \rightarrow \mathbb{C}$ where I is an open interval in $\mathbb{R}$ and $\xi = \operatorname{Re} \gamma$ and $\eta = \operatorname{Im} \gamma$ satisfy $\xi, \eta \in C^0(I)$, we define for $a, b \in I$ with $a < b$

$$\int_a^b \gamma(t) dt = \int_a^b \xi(t) dt + i \int_a^b \eta(t) dt.$$

Note that while this integral is complex valued, it is not a complex integral. We will come to complex integration later in the course.

Definition 3 Given $\gamma = \xi + i\eta : I \rightarrow \mathbb{C}$ where I is an open interval in $\mathbb{R}$ and $\xi = \operatorname{Re} \gamma$ and $\eta = \operatorname{Im} \gamma$ satisfy $\xi, \eta \in C^1(I)$, we define

$$\frac{d\gamma}{dt} = \xi' + i\eta'$$

as in Assignment 5. Note that while this derivative is complex valued, it is not a complex derivative as considered in Assignment 1.

Problem 4 Consider $\gamma = \xi + i\eta : I \rightarrow \mathbb{C}$ where I is an open interval in $\mathbb{R}$ and $\xi = \operatorname{Re} \gamma$ and $\eta = \operatorname{Im} \gamma$ satisfy $\xi, \eta \in C^1(I)$. Show that for $a, b \in I$ with $a < b$

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a).$$

Hint: Use the appropriate definitions and results from one dimensional calculus/calc I.

Problem 5 (a chain rule) Given $\gamma = \xi + i\eta : I \rightarrow \mathbb{C}$ with $\xi, \eta \in C^1(I)$ as in Problem 4 and $f : \mathbb{C} \rightarrow \mathbb{C}$ complex differentiable as in Problem 6 of Assignment 1, formulate an appropriate **chain rule** for the composition

$$f \circ \gamma : I \rightarrow \mathbb{C}$$

and verify this chain rule.

Problem 6 Recall the ODE (3) from Problem 3 above. Assume $\gamma(0) \notin \mathbb{R}$. Use the chain rule from Problem 5 to solve the IVP

$$\begin{cases} \frac{d\gamma}{dt} = \gamma^2, & t \in \mathbb{R} \\ \gamma(0) = w \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \quad (4)$$

Problem 7 Explain why the real solutions of (3) are not defined for all $t \in \mathbb{R}$ but the solutions of (4) have solutions with interval of existence $\mathbb{R}$.

Problem 8 (complex exponential) Define the complex exponential function

$$\exp : \mathbb{C} \rightarrow \mathbb{C} \quad \text{with} \quad \exp(z) = e^z.$$

(a) Consider $\Omega = \{z \in \mathbb{C} : 0 \leq \text{Im}(z) < 2\pi\}$. Draw Ω and find/draw the image of

$$\exp|_{\Omega}.$$

(b) Find an appropriate Riemann surface $\mathcal{R}$ for the complex exponential

$$\exp : \mathbb{C} \rightarrow \mathcal{R} = \bigcup \Sigma_j.$$

(c) Give a formula for the inverse

$$\left(\exp|_{\Omega_j} \right)^{-1} : \Sigma_j \rightarrow \Omega_j$$

where Σ_j is an appropriate sheet of your Riemann surface from part (b) above (with interior diffeomorphic to $\mathbb{C}$) and

$$\Omega_j = \left(\exp|_{\Omega_j} \right)^{-1} (\Sigma_j).$$

The function you have defined is called a **branch of the complex logarithm**.

(d) Show the function $\exp : \mathbb{C} \rightarrow \mathbb{C}$ you defined above is an entire function and find the complex derivative

$$\frac{d}{dz} e^z.$$

Hint: The definition of the complex derivative is/was given in Assignment 1.

Given an open interval $I \subset \mathbb{R}$ and a function $\gamma : I \rightarrow \mathbb{C}$, we say $\gamma \in C^1(I \rightarrow \mathbb{C})$, that is γ is **continuously differentiable** if

$$\xi = \operatorname{Re} \gamma : I \rightarrow \mathbb{R} \quad \text{and} \quad \eta = \operatorname{Im} \gamma : I \rightarrow \mathbb{R}$$

satisfy $\xi, \eta \in C^1(I)$. In this case, we write

$$\frac{d\gamma}{dt} = \xi' + i\eta'$$

as in Definition 3 above. More generally, if $\gamma : I \rightarrow \mathbb{C}^n$ with $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, we say $\gamma \in C^1(I \rightarrow \mathbb{C}^n)$ if $\gamma_j \in C^1(I \rightarrow \mathbb{C})$ for $j = 1, 2, \dots, n$.

In this case we write

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \dots, \frac{d\gamma_n}{dt} \right).$$

Thus, a typical system of first order ODEs for $\gamma \in C^1(I \rightarrow \mathbb{C}^n)$ has the form

$$\frac{d\gamma}{dt} = F(\gamma, t)$$

where $F : \Omega \times I \rightarrow \mathbb{C}^n$ with Ω an open subset of $\mathbb{C}^n$ containing the image of γ .

Problem 9 Given $n \in \mathbb{N}$ and $v = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n$ use your chain rule from Problem 5 to compute/show

$$\frac{d}{dt} [e^{\lambda t} v] = \lambda e^{\lambda t} v$$

where $\lambda \in \mathbb{C}$ is a fixed complex scalar/constant.

Definition 4 (linear system of ODEs, real valued case) Given $n \in \mathbb{N}$, an open interval $I \subset \mathbb{R}$, and n^2 functions $a_{jk} \in C^\infty(I)$, $j, k = 1, 2, \dots, n$, a **first order linear differential operator** $\Lambda : C^\infty(I \rightarrow \mathbb{R}^n) \rightarrow C^\infty(I \rightarrow \mathbb{R}^n)$ is an operator with values of the form

$$\Lambda[\mathbf{x}] = \sum_{j=1}^n \left[x'_j - \left(\sum_{k=1}^n a_{jk} x_k \right) \right] \mathbf{e}_j \quad (5)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $x_j \in C^\infty(I)$ for $j = 1, 2, \dots, n$.

A linear function $\Lambda : C^1(I \rightarrow \mathbb{R}^n) \rightarrow C^0(I \rightarrow \mathbb{R}^n)$ with values given by (5) where $a_{jk} \in C^0(I)$ for $j, k = 1, 2, \dots, n$ is also sometimes called a first order linear differential operator.

A **linear system of ODEs** for $\mathbf{x} \in C^1(I \rightarrow \mathbb{R}^n)$ is a system of ODEs having the form

$$\Lambda[\mathbf{x}] = F(t) \quad (6)$$

where $F \in C^0(I \rightarrow \mathbb{R}^n)$.

The system is said to be **homogeneous** if $F \equiv \mathbf{0} \in \mathbb{R}^n$.

Theorem 2 (linear existence and uniqueness) If I is an open interval of $\mathbb{R}$ with

- (i) $t_0 \in I$,
- (ii) $a_{jk} \in C^0(I)$ for $j, k = 1, 2, \dots, n$, and
- (iii) $F \in C^0(I \rightarrow \mathbb{R}^n)$,

then for any $\mathbf{y} \in \mathbb{R}^n$ the initial value problem

$$\begin{cases} \Lambda[\mathbf{x}] = F(t), & t \in I, \quad j = 1, 2, \dots, n \\ \mathbf{x}(t_0) = \mathbf{y} \end{cases} \quad (7)$$

has a unique solution $\mathbf{x} \in C^1(I \rightarrow \mathbb{R}^n)$.

Note carefully the interval of existence is I .

Note there is no requirement on the initial value $\mathbf{y} \in \mathbb{R}^n$.

Problem 10 Consider the system of ODEs

$$\frac{d\gamma}{dt} = \sum_{j=1}^n \left(\sum_{k=1}^n a_{jk}(t) \gamma_k \right) \mathbf{e}_j \quad (8)$$

where $a_{jk} \in C^0(I \rightarrow \mathbb{C})$ for some open interval $I \subset \mathbb{R}$ and $j, k = 1, 2, \dots, n$.

(a) Formulate (and prove) an existence and uniqueness theorem for the IVP

$$\begin{cases} \frac{d\gamma}{dt} = \sum_{j=1}^n \left(\sum_{k=1}^n a_{jk}(t) \gamma_k \right) \mathbf{e}_j & t \in I \\ \gamma(t_0) = \zeta \end{cases} \quad (9)$$

for (the coordinate functions of) a complex vector valued function $\gamma \in C^1(I \rightarrow \mathbb{C}^n)$. Hint: Write $\gamma = (\xi_1 + i\eta_1, \xi_2 + i\eta_2, \dots, \xi_n + i\eta_n)$ and $a_{jk} = \alpha_{jk} + i\beta_{jk}$ for $\xi_j, \eta_j, \alpha_{jk}, \beta_{jk} \in C^0(I)$.

(b) Under what conditions is the (real) linear system (6) autonomous?

(c) In the case a_{jk} is a constant complex number for $j, k = 1, 2, \dots, n$, find a solution of (8). Hint(s): Define an appropriate linear operator $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$. An operator on a complex linear space has an eigenvector.

(d) In the case where a_{jk} is a constant complex number for $j, k = 1, 2, \dots, n$ and L admits an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ of eigenvectors, find all solutions of (9).