

Assignment 5: ODE: Existence and Uniqueness (single ODEs)

Due Thursday October 15, 2026

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Problem 1 Consider the ordinary differential equation (ODE)

$$x'(t) = \sin(t^2) \tag{1}$$

for a function $x \in C^1(\mathbb{R})$.

- (a) Draw the graph of $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(t) = \sin(t^2)$. Use mathematical software if necessary.
- (b) Solve (1).
 - (i) Draw a picture to illustrate your solution from part (b).
 - (ii) Illustrate the value of $x = x(t)$ in your drawing/picture from part (a).
- (c) Solve the initial value problem (IVP)

$$\begin{cases} x' = \sin(t^2), \\ x(\pi/4) = 1. \end{cases}$$

- (i) What is the largest interval I on which your solution from part (c) is valid?
- (ii) Find $x(0)$ and draw the graph of your solution from part (c).

Problem 2 Consider $f : (-1, 1) \rightarrow \mathbb{R}$ by

$$f(t) = \frac{1}{\sqrt{1-t^2}}.$$

- (a) Draw the graph of f . Use mathematical software if necessary.

Let $I = (a, b)$ be an appropriate open interval with $a < 0 < b$ and consider the IVP

$$\begin{cases} x' = \frac{1}{\sqrt{1-t^2}}, & t \in I \\ x(0) = 0. \end{cases} \quad (2)$$

(b) Let I denote the largest possible interval on which you think you might be able to find a solution $x \in C^1(I)$ for (2). What is this largest possible interval I ?

(i) Without solving the IVP (2) use the ODE to give an explicit estimate for $x(t)$ giving a fixed number $M > 0$ for which $|x(t)| \leq M$ for $x \in I$. Hint(s):

$$x(t) - x(0) = \int_0^t x'(\tau) d\tau$$

and for $\tau > 0$ there holds $1 - \tau^2 > 1 - \tau$.

(ii) Given a solution $x \in C^1(I)$, let

$$m = \inf\{M > 0 : x(t) < M, t \in I\}.$$

Show $x : I \rightarrow (-m, m)$ is an odd bijection.

(c) Solve the IVP (2).

(i) Draw the graph of your solution $x = x(t)$ from part (c).

(ii) On what open interval $I = (a, b)$ is your solution valid.

(d) Let $\phi : (-m, m) \rightarrow I$ denote the inverse of your solution x .

(i) Find extensions $x_* \in C^0[a, b]$ and $\phi_* \in C^0[-m, m]$ of your solution x and its inverse ϕ respectively where $[a, b]$ is the closure $\bar{I}$ of your interval of existence. Hint: You need to define x_* appropriately at $t = a$ and $t = b$. Monotonicity may help.

(ii) Evaluate the assertion $x_* \in C^1(\bar{I})$.

(iii) Evaluate the assertion $\phi_* \in C^1[-m, m]$.

(iv) Compute $\phi'(\xi)$ and $\phi''(\xi)$ in terms of $\phi(\xi)$ for $\xi \in (-m, m)$. Hint(s): For the first computation start with the identity $x \circ \phi = \text{id}_{(-m, m)}$ and use the chain rule. Differentiate again using the chain rule and your expression for ϕ' .

Problem 3 Recall the C^0 norm is given by

$$\begin{aligned}\|u\|_{C^0} &= \sup\{|u(t)| : t \in I\} && \text{for an open interval } I \text{ and } u \in C^0(I), \text{ and} \\ \|u\|_{C^0} &= \max\{|u(t)| : t \in \bar{I}\} && \text{for a closed interval } \bar{I} \text{ and } u \in C^0(\bar{I}).\end{aligned}$$

Determine the following C^0 norms:

- (a) $\|x\|_{C^0(I)}$ with x your solution from part (c) of Problem 2.
- (b) $\|\phi\|_{C^0(-m,m)}$ with ϕ the inverse of your solution from part (c) of Problem 2.
- (c) $\|\phi_*\|_{C^0[-m,m]}$ with ϕ_* the extension of the inverse of your solution from part (c) of Problem 2.

Theorem 1 (local existence and uniqueness for a single ODE for a real valued function) Given open intervals I and J with $(x_0, t_0) \in J \times I$, if $F : J \times I \rightarrow \mathbb{R}$ satisfies

$$F \in C^0(J \times I)$$

and

$$\frac{\partial F}{\partial x} \in C^0(J \times I)$$

where

$$\frac{\partial F}{\partial x} : J \times I \rightarrow \mathbb{R} \quad \text{by} \quad \frac{\partial F}{\partial x}(x, t) = \frac{\partial}{\partial x} F(x, t),$$

then there exists some $\delta > 0$ so that the IVP

$$\begin{cases} x' = F(x, t), & t \in (t_0 - \delta, t_0 + \delta) \\ x(t_0) = x_0 \end{cases} \quad (3)$$

has a unique solution $x \in C^1(t_0 - \delta, t_0 + \delta)$.

The function F is called the **structure function** for the ODE $x' = F(x, t)$.

An ODE of the form $x' = F(x, t)$ is said to be **regular**. A more general ODE might have the form $G(x', x, t) = 0$ where $G : H \times J \times I \rightarrow \mathbb{R}$.

Problem 4 What does Theorem 1 tell you about the ODEs/IVPs in Problems 1 and 2? (Apply Theorem 1 to these problems.)

Definition 1 Given open intervals I and J with $(x_0, t_0) \in J \times I$ and a structure function $F \in C^0(J \times I)$, the **maximum existence interval** I_0 , if such an interval exists, is the union of all open intervals U for which the IVP

$$\begin{cases} x' = F(x, t), & t \in U \\ x(t_0) = x_0 \end{cases} \quad (4)$$

has a solution $x \in C^1(U)$.

Problem 5 (a) What is the maximum existence interval for the IVP in Problem 1 above?

(b) What is the maximum existence interval for the IVP (2) in Problem 2 above?

(c) Use Theorem 1 to guess any and all maximum existence interval associated with an IVP for

$$x' = \frac{t - x}{2t + 5x}.$$

(Explain where and why Theorem 1 may be applied to an IVP for this ODE and apply Theorem 1 to that problem. Draw the open region $U \subset \mathbb{R}^2$ consisting of initial values/points (x_0, t_0) where Theorem 1 may be applied.)

(d) Use Theorem 1 to guess any and all maximum existence interval associated with an IVP for

$$x' = \sqrt{1 - t^2 - x^2}.$$

(Explain where and why Theorem 1 may be applied to an IVP for this ODE and apply Theorem 1 to that problem. Draw the open region $U \subset \mathbb{R}^2$ consisting of initial values/points (x_0, t_0) where Theorem 1 may be applied.)

Problem 6 Prove Theorem 1 in the special case where $F(x, t) = f(t)$ for some $f \in C^0(I)$.

Problem 7 Consider the ODE

$$x' = x^2. \tag{5}$$

(a) Solve (5).

(b) Solve the IVP

$$\begin{cases} x' = x^2, & t \in \mathbb{R}. \\ x(t_0) = x_0. \end{cases} \tag{6}$$

(c) Apply Theorem 1 to (6).

(d) What is the maximum interval of existence.

(e) Fix (6). (If you don't see a problem, think more carefully.)

Problem 8 Consider the ODE

$$x' = \sqrt{|x|}. \tag{7}$$

(a) Find three different solutions $x \in C^1(-1, 1)$ for the IVP

$$\begin{cases} x' = \sqrt{|x|}, & t \in \mathbb{R}. \\ x(0) = 0. \end{cases} \tag{8}$$

(b) Find the maximum interval of existence for each solution you found in part (a) above.

(c) What does Theorem 1 tell you about (8).

Given an open interval $I \subset \mathbb{R}$ and a function $\gamma : I \rightarrow \mathbb{C}$, we say $\gamma \in C^1(I \rightarrow \mathbb{C})$ if

$$\xi = \operatorname{Re} \gamma : I \rightarrow \mathbb{R} \quad \text{and} \quad \eta = \operatorname{Im} \gamma : I \rightarrow \mathbb{R}$$

satisfy $\xi, \eta \in C^1(\mathbb{R})$. In this case, we write

$$\frac{d\gamma}{dt} = \xi' + i\eta'.$$

More generally, if $\gamma : I \rightarrow \mathbb{C}^n$ with $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, we say $\gamma \in C^1(I \rightarrow \mathbb{C}^n)$ if $\gamma_j \in C^1(I \rightarrow \mathbb{C})$ for $j = 1, 2, \dots, n$.

Thus, a typical (single regular first order) ODE for $\gamma \in C^1(I \rightarrow \mathbb{C})$ has the form

$$\frac{d\gamma}{dt} = F(\gamma, t)$$

where $F : \Omega \times I \rightarrow \mathbb{C}$ with Ω an open subset of $\mathbb{C}$ containing the image of γ .

Problem 9 Draw the image of the function $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ with values given by the following formulas:

(a) $\gamma(t) = t(1 + i)$.

(b) $\gamma(t) = t + i$.

(c) $\gamma(t) = 1 + it$.

(d) $\gamma(t) = e^{t+i}$.

(e) $\gamma(t) = e^{1+it}$.

Problem 10 Consider the ODE

$$\frac{d\gamma}{dt} = \gamma^2. \tag{9}$$

(a) If $\gamma = \xi + i\eta$ with $\xi = \operatorname{Re} \gamma$ and $\eta = \operatorname{Im} \gamma$ as above, what does the ODE (9) say about ξ and η ?

(b) What does Theorem 1 say about ξ and η ?

(c) Can you solve (9)?

(d) Use mathematical software to draw the image of the solution of the IVP

$$\begin{cases} \frac{d\gamma}{dt} = \gamma^2, & t \in \mathbb{R} \\ \gamma(0) = i. \end{cases} \tag{10}$$