

Extra Problems on Linear Operators

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Problem 1 Consider the operators $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \geq 3$ and values given as follows:

(a) $L(z_1, z_2, \dots, z_n) = (2z_2, 0, 5z_3, 0, 0, 0, \dots, 0).$

(b) $L(z_1, z_2, \dots, z_n) = (z_1, 2z_2, 3z_3, \dots, nz_n).$

In each case, find all eigenvalues, eigenvectors and invariant subspaces of L .

Problem 2 (projection operator) Consider $p : E_1 \oplus E_2 \rightarrow E_1 \oplus E_2$ by

$$p(v_1 + v_2) = v_1 \quad \text{for} \quad v = v_1 + v_2 \in E_1 \oplus E_2 \quad \text{with} \quad v_j \in E_j, \quad j = 1, 2.$$

Find the eigenvalues and eigenvectors of p . (You may assume E_j is a subspace of the linear space $W = E_1 \oplus E_2$ and $E_j \neq \{\mathbf{0}\}$ for $j = 1, 2$.)

Problem 3 Find an operator $L \in \mathcal{L}(\mathbb{R}^4)$ with no eigenvalues.

Problem 4 If $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$ and

$$L(z_1, z_2, \dots, z_n) = (z_1 + z_2 + \dots + z_n, z_1 + z_2 + \dots + z_n, \dots, z_1 + z_2 + \dots + z_n),$$

find all eigenvalues and eigenvectors of L .

Problem 5 Given an eigenvalue $\lambda \in \mathbb{C}$ for an **invertible** operator $L \in \mathcal{L}(\mathbb{C}^n)$, find an eigenvalue and eigenvector for $L^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$.

Problem 6 True or False: There exists an invertible operator $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with a zero eigenvector.

Problem 7 Assume $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$ and

$$\begin{cases} L(v) = 3w \\ L(w) = 3v \end{cases}$$

for some pair of nonzero vectors $v, w \in \mathbb{C}^n$. Show some integer multiple of 3 is an eigenvalue.

Problem 8 If $L, M \in \mathcal{L}(\mathbb{C}^n)$ with $n \geq 1$, then show $L \circ M$ and $M \circ L$ have the same eigenvalues.

Polynomial of an operator: Given a polynomial $P \in \mathbb{C}[X]$ with

$$P = \sum_{j=0}^k a_j X^j$$

and an operator $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$, the linear operator

$$p(L) = a_0 \operatorname{id}_{\mathbb{C}^n} + \sum_{j=1}^k a_j L^j$$

where $L^j = L \circ L \circ L \circ \cdots \circ L$ (the composition of L with itself j times) is called the polynomial P **evaluated** on L . Note: One can also evaluate a polynomial on elements of $\mathbb{C}$ to obtain a polynomial function, but this evaluation on an operator is something different.

Problem 9 If $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$, then verify

$$\operatorname{id}_{\mathbb{C}^n} - L \circ L = (\operatorname{id}_{\mathbb{C}^n} - L) \circ (\operatorname{id}_{\mathbb{C}^n} + L) = (\operatorname{id}_{\mathbb{C}^n} + L) \circ (\operatorname{id}_{\mathbb{C}^n} - L).$$

Problem 10 If $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$ and $L^n = L \circ L \circ \cdots \circ L$ (n times) is the constant zero operator, then express $(L - \operatorname{id}_{\mathbb{C}^n})^{-1}$ as a polynomial in L . (You should also justify the assumption that $L - \operatorname{id}_{\mathbb{C}^n}$ is invertible.)

Problem 11 If $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$ and

(i) $L \circ L = \text{id}_{\mathbb{C}^n}$ and

(ii) -1 is not an eigenvalue of L ,

then (show) $L = \text{id}_{\mathbb{C}^n}$.

Problem 12 If $L \in \mathcal{L}(\mathbb{C}^n)$ with $n \in \mathbb{N}$ and $L \circ L = L$, then (show)

$$\mathbb{C}^n = \mathcal{N}(L) \oplus L(\mathbb{C}^n).$$

Problem 13 If $L \in \mathcal{L}(W)$ where W is a linear space and W has a basis consisting of eigenvectors, then (show)

$$W = \mathcal{N}(L) \oplus L(W).$$

Problem 14 Consider $L \in \mathcal{L}(\mathbb{C}[X])$ by

$$L\left(\sum_{j=0}^k a_j X^j\right) = \sum_{j=1}^k j a_j X^{j-1} = P'.$$

(a) Find the null space $\mathcal{N}(L)$.

(b) Find the range $L(\mathbb{C}[X])$.

(c) Find the sum $\mathcal{N}(L) + L(\mathbb{C}[X])$.

(d) Is the sum you found in part (c) a direct sum?

Problem 15 (direct sum of eigenspaces) Recall that given an eigenvalue λ for $L \in \mathcal{L}(W)$ where W is a linear space, the eigenspace associated with λ is

$$\mathcal{N}(L - \lambda \text{ id}) = \{v \in W : L(v) = \lambda v\}.$$

Assume $\lambda_1, \lambda_2, \dots, \lambda_k$ are **distinct** eigenvalues for $L \in \mathcal{L}(W)$. Then (show)

$$\mathcal{N}(L - \lambda_1 \text{ id}) + \mathcal{N}(L - \lambda_2 \text{ id}) + \cdots + \mathcal{N}(L - \lambda_k \text{ id})$$

is a direct sum.

Hint(s): Given subspaces $E_1, E_2, \dots, E_k \subset W$ the sum

$$E_1 + E_2 + \cdots + E_k = \{v_1 + v_2 + \cdots + v_k : v_j \in E_j, j = 1, 2, \dots, k\}$$

is always a subspace. What it means to be a **direct sum** is that for every element $v = v_1 + v_2 + \cdots + v_k$ in the sum the projections v_j are unique, i.e., there is only one choice of $v_j \in E_j$ for which $v = v_1 + v_2 + \cdots + v_k$ with $v_j \in E_j$, $j = 1, 2, \dots, k$. If there is more than one way to write an element $v = v_1 + v_2 + \cdots + v_k$ in the sum, then there is more than one way to write the zero element $\mathbf{0}$ in the sum.

Problem 16 Assume that for every $\nu \in \mathbb{C}$, there holds

$$W = \mathcal{N}(L - \nu \text{ id}) \oplus \text{range}(L - \nu \text{ id})$$

where $L \in \mathcal{L}(W)$ with W a finite dimensional complex linear space and $\text{range}(M) = M(W)$ is the range of a linear operator $M \in \mathcal{L}(W)$. Show there exists a basis for W consisting of eigenvectors of L .

Hint(s): For most values of ν there must hold $\mathcal{N}(L - \nu \text{ id}) = \{\mathbf{0}\}$. Do you know why?

Problem 17 Assume $\{v_1, v_2, \dots, v_n\}$ is a basis for a complex linear space W consisting of eigenvectors $v_1, v_2, \dots, v_n$ for a linear operator $L \in \mathcal{L}(W)$. Note: Distinct basis vectors here may correspond to the same eigenvalue.

Show that for every $\nu \in \mathbb{C}$, there holds

$$W = \mathcal{N}(L - \nu \text{ id}) \oplus \text{range}(L - \nu \text{ id}).$$

Problem 18 Assume $\{v_1, v_2, \dots, v_n\}$ is a basis for a complex linear space W consisting of eigenvectors $v_1, v_2, \dots, v_n$ for a linear operator $L \in \mathcal{L}(W)$. Assume furthermore that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with

$$L(v_j) = \lambda_j v_j, \quad j = 1, 2, \dots, n$$

are distinct.

If $M \in \mathcal{L}(W)$ and

$$M(v_j) = \mu_j v_j, \quad j = 1, 2, \dots, n$$

for some $\mu_1, \mu_2, \dots, \mu_n \in \mathbb{C}$, then (show)

$$L \circ M = M \circ L.$$