Assignment 4: Linear Operators
Due Thursday October 8, 2026

John McCuan

In the problems below V and X are linear spaces over the same field F and $\mathcal{L}(X)$ denotes the linear space of automorphisms/operators $\mathcal{L}(X \to X)$. Roughly speaking, problems involving the space X are valid for all linear spaces (finite or infinite dimensional), and problems involving the space V concern a finite dimensional linear space.

Problem 1 **(a)** If $L, M \in \mathcal{L}(X)$, then

$$M \circ L \text{ injective} \qquad \text{implies} \qquad M \text{ and } L \text{ are both injective.}$$

(b) If V is finite dimensional and $L, M \in \mathcal{L}(V)$, then

$$M \circ L \text{ surjective} \qquad \text{implies} \qquad M \text{ and } L \text{ are both surjective.}$$

(c) Let $\mathbb{C}^\infty$ denote the collection of sequences of complex numbers, so an element of $\mathbb{C}^\infty$ may be denoted $(z_1, z_2, z_3, \ldots)$ where $z_j \in \mathbb{C}$ for $j = 1, 2, 3, \ldots$. Consider $M \in \mathcal{L}(\mathbb{C}^\infty)$ by

$$M(z_1, z_2, z_3, \ldots) = (z_2, z_3, z_4, \ldots).$$

Find a nonsurjective linear operator $L \in \mathcal{L}(\mathbb{C}^\infty)$ for which

$$M \circ L = \mathrm{id}_{\mathbb{C}^\infty}.$$

Problem 2 If V is finite dimensional and $L, M \in \mathcal{L}(V)$ satisfy

$$M \circ L = \mathrm{id}_V,$$

that is, L is a right inverse of M and M is a left inverse of L, then (show)

$$L \circ M = \mathrm{id}_V,$$

so that $L, M \in \mathcal{L}^\times(V)$ the group of invertible operators in the ring $\mathcal{L}(V)$.

Problem 3 If V is finite dimensional and $L, M, N \in \mathcal{L}(V)$ with

$$L \circ M \circ N = \mathrm{id}_V,$$

then (show) $M \in \mathcal{L}^\times(V)$ and $M^{-1} = N \circ L$.

Problem 4 Given $L \in \mathcal{L}(X)$ a subspace $E \subset X$ is said to be and **invariant subspace** if the range of the restriction

$$L\Big|_E : E \to X$$

is a subset of E. Show range$(L) = L(X)$ is an invariant subspace.

Given a complex linear space X, an **eigenvector** for $L \in \mathcal{L}(X)$ is a complex number λ for which there exists a nonzero $v \in X\backslash\{\mathbf{0}\}$ satisfying

$$L(v) = \lambda v. \tag{1}$$

When λ is an eigenvalue, any "corresponding" vector $v \in X\backslash\{\mathbf{0}\}$ for which (1) holds is called an **eigenvector**. Also, given an eigenvector λ the set

$$\{v \in X : L(v) = \lambda v\}$$

is called the **eigenspace** associated with the eigenvalue λ. Note: Not every element of the eigenspace is an eigenvector.

Problem 5 Given an eigenvalue $\lambda \in \mathbb{C}$ for $L \in \mathcal{L}(X)$, why isn't every vector in the eigenspace

$$E_\lambda = \{v \in X : L(v) = \lambda v\}$$

an eigenvector for L?

Problem 6 Consider $L \in \mathcal{L}(\mathbb{C}^2)$ with

$$L(z_1, z_2) = (-z_2, z_1).$$

(a) $(1, i) \in \mathbb{C}^2$ is an eigenvector for L. Find the eigenvalue λ and the associated eigenspace E_λ.

(b) Find a different eigenvalue for L.

Problem 7 Consider the operators $L \in \mathcal{L}(\mathbb{C}^2)$ with values given as follows:

(a) $L(z_1, z_2) = (z_2, 0)$.

(b) $L(z_1, z_2) = (-3z_2, z_1)$.

In each case, find all eigenvalues and eigenvectors for L.

Problem 8 Consider the linear operators $L \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ with values given as follows:

(a) $L(p) = p'$.

(b) $L(p) = zp'$.

In each case, find all eigenvalues and eigenvectors for L. (Here p' denotes the complex derivative of the polynomial (function) p and "z" in part **(b)** is indicating the product of p' with the identity function on $\mathbb{C}$, that is the identity with respect to composition $\mathrm{id} \in \mathcal{P}(\mathbb{C})$ with $\mathrm{id}(z) = z$.)

Problem 9 Suppose $L \in \mathcal{L}(\mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ are given. Is there some $\mu \in \mathbb{C}$ for which

(i) $|\mu - \lambda| < 1$ and

(ii) $L - \mu\,\mathrm{id} : \mathbb{C}^2 \to \mathbb{C}^2$ is a bijection, i.e., invertible?

How do you know?

Problem 10 The Fibonacci sequence $a_1, a_2, a_3, \dots$ is given inductivly by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$. Consider $L \in \mathcal{L}(\mathbb{R}^2)$ by

$$L(x_1, x_2) = (x_2, x_1 + x_2).$$

(a) Show that for each positive integer n there holds

$$L \circ L \circ \cdots \circ L(0, 1) = L^n(0, 1) = (a_n, a_{n+1}).$$

(b) Find the eigenvalues of L.

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of L.

(d) Verify that for each positive integer n there holds

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

(e) Verify that for each positive integer n the Fibonacci number a_n is the integer closest to

$$\left(\frac{1 + \sqrt{5}}{2} \right)^n.$$