Assignment 3: Linear Functions (basics)
Due Thursday September 24, 2026

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In the problems below V, W, X, Y are linear spaces over the same field F and $\mathcal{L}(X \to Y)$ denotes the collection of linear functions with domain X and codomain Y. Roughly speaking, problems involving the spaces X and Y are valid for all linear spaces (finite or infinite dimensional), and problems involving the spaces V and W concern finite dimensional linear spaces.

Problem 1 **(a)** If $L \in \mathcal{L}(X \to Y)$ and $a \in F$, then the scaling

$$aL : X \to Y \qquad \text{by} \qquad (aL)(v) = a\ L(v)$$

satisfies $aL \in \mathcal{L}(X \to Y)$. That is, there is a scaling of linear functions using scalars in F.

(b) If $L, M \in \mathcal{L}(X \to Y)$ show the sum

$$M + L : X \to Y \qquad \text{by} \qquad (M + L)(v) = M(v) + L(v)$$

satisfies $M + L \in \mathcal{L}(X \to Y)$. That is, a sum of linear functions is linear.

(c) If $L \in \mathcal{L}(X \to Y)$ and $M \in \mathcal{L}(Y \to Z)$, show the composition

$$M \circ L : X \to Z \qquad \text{by} \qquad M \circ L(v) = M(L(v))$$

satisfies $M \circ L \in \mathcal{L}(X \to Z)$. That is, a composition of linear functions is linear.

Problem 2 **(a)** Show $\mathcal{L}(X \to Y)$ is a linear space over F with the scaling and addition described in parts **(a)** and **(b)** of Problem 1 above.

(b) Show $\mathcal{L}(X) = \mathcal{L}(X \to X)$ is a ring with "multiplication" given by function composition.

Problem 3 If $L \in \mathcal{L}(X \to Y)$, show

$$L(\mathbf{0}_X) = \mathbf{0}_Y \in Y.$$

Problem 4 For this problem, consider $V = \mathbb{C}^2$ and $W = \mathbb{C}$ as linear spaces over the field $\mathbb{C}$.

(a) Give an example of a function $f : \mathbb{C}^2 \to \mathbb{C}$ for which

$$f(\alpha v) = \alpha f(v) \qquad \text{whenever} \qquad \alpha \in \mathbb{C} \quad \text{and} \quad v \in \mathbb{C}^2$$

but f is **not** linear, i.e., $f \notin \mathcal{L}(\mathbb{C}^2 \to \mathbb{C})$.

(b) Give an example of a function $f : \mathbb{C}^2 \to \mathbb{C}$ for which

$$f(v + w) = f(v) + f(w) \qquad \text{whenever} \qquad v, w \in \mathbb{C}^2$$

but f is **not** a linear function.

Problem 5 If $L \in \mathcal{L}(X \to Y)$ and $\{L(v_1), L(v_2), \ldots, L(v_k)\}$ is a linearly independent set in $L(V) \subset Y$, show $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set in X.

Problem 6 Given $L \in \mathcal{L}(X \to Y)$, show the following

(a) $\mathcal{N}(L) = \{v \in X : L(v) = 0\}$ is a subspace of X. (This subspace is called the **null space**.)

(b) $L(X) = \{L(v) : v \in X\}$ is a subspace of Y. This subspace is (of course) called the range.

Problem 7 Assume $L \in \mathcal{L}(V \to Y)$ where V is a finite dimensional linear space. Assume $\{\nu_1, \nu_2, \ldots, \nu_k\}$ is a basis for $\mathcal{N}(L)$ and $\{\nu_1, \nu_2, \ldots, \nu_k, v_1, v_2, \ldots, v_m\}$ is a basis for V.

(a) Show $\text{span}\{L(v_1), L(v_2), \ldots, L(v_m)\} = L(V)$, so $L(V)$ is finite dimensional.

(b) Show $\{L(v_1), L(v_2), \ldots, L(v_m)\}$ is a linearly independent set in Y.

Your solution of this problem implies the **fundamental theorem of dimension**:

$$\dim \mathcal{N}(L) + \dim L(V) = \dim V.$$

Problem 8 Assume V is a finite dimensional linear space and $\{v_1, v_2, \ldots, v_k\}$ is a subset of V. Consider the linear function $L \in \mathcal{L}(F^k \to V)$ by

$$L(a_1, a_2, \ldots, a_k) = \sum_{j=1}^{k} a_j v_j.$$

(a) If $\text{span}\{v_1, v_2, \ldots, v_k\} = V$, categorize the following assertions as "true" or "false." Explain your answers in all cases.

(i) L is injective. (True or false?)

(ii) L Is surjective. (True or false?)

(iii) $k \leq \dim(V)$. (True or false?)

(a) If $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set, categorize the following assertions as "true" or "false." Explain your answers in all cases.

(i) L is injective. (True or false?)

(ii) L Is surjective. (True or false?)

(iii) $k \leq \dim(V)$. (True or false?)

Problem 9 (continuity) For this problem we assume F is a field with a well-defined **absolute value**. For example, $F = \mathbb{R}$ or $F = \mathbb{C}$ are both fields with an absolute value. A linear space X is said to be a **normed space** if there is a function $\|\cdot\| : X \to [0, \infty)$ satisfying

(N1) If $\|v\| = 0$, then $v = \mathbf{0}_X \in X$.

(N2) $\|av\| = |a|\ \|v\|$ for all $a \in F$ and $v \in X$.

(N3) $\|v + w\| \leq \|v\| + \|w\|$ whenever $v, w \in X$.

Elements of a normed space are called **vectors**. Each such element has a well-defined **magnitude** $\|v\|$.

(a) Condition **(N1)** is called the **positive definite** condition. Show $\|\mathbf{0}_X\| = 0$. Hint: Remember $\mathbf{0}_X = 0v$ for every $v \in X$.

Condition **(N2)** is called **nonnegative homogeneiety**, and condition **(N3)** is called the triangle inequality. In the same way a group is not just a set, but a set with an operation, the linear space X is not a normed space by itself, but rather a normed space is a pair $(X, \|\ \cdot\ \|)$ consisting of the linear space X and a given norm.

(b) Given normed spaces X and Y a function $f : X \to Y$ is **continuous** if for any $v \in X$ and any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\|f(w) - f(v)\|_Y \leq \epsilon \qquad \text{if} \qquad \|w - v\|_X < \delta.$$

Show that if $L \in \mathcal{L}(X \to Y)$ is continuous, then there exists some $M > 0$ so that

$$\|L(v)\|_Y \leq M\|v\|_X \qquad \text{for all } x \in X. \tag{1}$$

Hint: Take $v = \mathbf{0}_X$ in the definition of continuity and then use scaling.

(c) A linear function $L \in \mathcal{L}(X \to Y)$ satisfying (1) is said to be **bounded**. Show a bounded linear function from a normed space X to a normed space Y is continuous.

Note: An absolute value on a field is a norm on the field F when considered as a linear space over itself. In this case, the positive homogeneity condition **(N2)** reduces to $\|ab\| = \|a\|\ \|b\|$ or equivalently $|ab| = |a|\ |b|$ for $a, b \in F$.

Problem 10 (algebraic dual and dual function) An element $\phi : \mathcal{L}(X \to F)$ is called a **linear functional**. The collection

$$X^{\text{dual}} = \mathcal{L}(X \to F)$$

of all linear functionals is called the **algebraic dual** of X or in this context just the **dual space** of X.

(a) Recall that $X^{\text{dual}} = \mathcal{L}(X \to F)$ is a linear space over F. See for example Problem 2**(a)** above.

Given linear spaces X and Y and $L \in \mathcal{L}(X \to Y)$, consider the function

$$L^{\text{dual}} : Y^{\text{dual}} \to X^{\text{dual}} \qquad \text{by} \qquad L^{\text{dual}}(\psi) = \psi \circ L.$$

That is, given $\psi \in Y^{\text{dual}}$ one takes the value $L^{\text{dual}}(\psi) = \phi \in X^{\text{dual}}$ to be the linear functional $\phi : X \to F$ with values

$$\phi(v) = \psi \circ L(v).$$

Show L^{dual} is a linear function. That is, $L^{\text{dual}} \in \mathcal{L}(Y^{\text{dual}} \to X^{\text{dual}})$.

The function L^{dual} is called the (algebraic) **dual function** of the linear function $L \in \mathcal{L}(X \to Y)$. Sometimes other notations for L^{dual} are used including L^* and L', but sometimes L' and L^* can denote other things (i.e., other functions, like an adjoint or a continuous dual function or a derivative) so one needs to be careful.

(b) If V is finite dimensional, show V^{dual} is finite dimensional and calculate the dimension of V^{dual}. Hint: Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V, and consider the function $\phi_j : V \to F$ with values

$$\phi_j\left(\sum_{\ell=1}^{n} a_\ell v_\ell\right) = a_j. \tag{2}$$

(c) Let V and W be finite dimensional linear spaces and $L \in \mathcal{L}(V \to W)$. If bases

$$\{v_1, v_2, \dots, v_n\} \qquad \text{and} \qquad \{w_1, w_2, \dots, w_m\}$$

have been specified for V and W respectively, there is a “canonical” linear function

$$M : \mathcal{L}(V \to W) \to F^{m \times n}$$

where $F^{m\times n}$ is the linear space of $m\times n$ matrices with entries in F. The values of the canonical function M are determined by the fact that for each $j = 1, 2, \ldots n$, the value $L(v_j)$ is given by a unique linear combination

$$L(v_j) = \sum_{\ell=1}^{m} a_{j\ell} w_\ell.$$

Accordingly, we assign $M(L) = (a_{j\ell})$. The image $M(L)$ is said to be **the matrix of L with respect to the bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$.** Notice that without the specification of the bases for V and W, there is no (particular) matrix associated with $L \in \mathcal{L}(V \to W)$.

Assuming $A = (a_{j\ell})$ is the matrix of L with respect to $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$, find the matrix of L^{dual} with respect to the bases for W^{dual} and V^{dual} obtained by the procedure suggested in (2).

Hints: The basis for V^{dual} obtained using (2) is called the **dual basis** for V. Thus, one is looking for the matrix B of L^{dual} with respect to the dual bases for W^{dual} and V^{dual}.

Let ψ_j be one of the dual basis functionals for W^{dual} and write $L^{\text{dual}}(\psi_j)$ as a linear combination

$$\sum b_{j\ell}\phi_\ell.$$

(Pick the correct index and range of indices for the sum.) Evaluate this expression on a basis element v_r in order to isolate one of the entries $b_{j\ell}$. On the other hand, write

$$L(v_r) = \sum_{s=1}^{m} a_{rs} w_s,$$

and compute $[L^{\text{dual}}(\psi_j)](v_r)$ in a different way to get an expression in terms of A.

The result you get should allow you to express the matrix B for L^{dual} in terms of the entries in the matrix $A = (a_{j\ell})$.

There is one more obvious step or two in this construction (which I'm not including in the assignment). One could ask if the function $\Phi : \mathcal{L}(V \to W) \to \mathcal{L}(W^{\text{dual}} \to V^{\text{dual}})$ by

$$\Phi(L) = L^{\text{dual}}$$

is a linear function. Assuming Φ is linear, what would be the size of the matrix for Φ?

One point of this construction is that if one had good relations between the null space and range of a linear function $L \in \mathcal{L}(V \to W)$ and the null space and range of $L^{\text{dual}} \in \mathcal{L}(W^{\text{dual}} \to V^{\text{dual}})$, then one could use the fundamental theorem of dimension (Problem 7 above) to obtain the relatively non-obvious fact that the dimension of the span of the columns of a matrix in $\mathbb{C}^m_{\text{column}}$, that is the column rank, is the same as the dimension of the span of the rows in $\mathbb{C}^n$. See for example Chapter 3 of Sheldon Axler's book *Linear Algebra Done Right*.