Assignment 2: Linear Spaces
Due Thursday September 17, 2026

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Problem 1 Prove that in a linear space X over a field F there holds

$$0v = \mathbf{0}$$

where $0 \in F$ is the multiplicative identity in the field, $v \in X$ is any element, and $\mathbf{0} \in X$ is the zero element in the linear space.

Problem 2 Determine if the field $\mathbb{F}_2 = \{0, 1\}$ with two elements and the obvious addition $(1+1 = 0)$ and multiplication is algebraically closed. Give a full explanation for your answer. Hint: The only possible first order factors in the polynomial ring $\mathbb{F}_2[x]$ are $x - 0$ and $x - 1$.

Problem 3 Assume X is a linear space over a field F. Show that if $a \in F$ and $v \in X$ and

$$av = \mathbf{0} \in X,$$

then either $a = 0 \in F$ for $v = \mathbf{0} \in X$. Hint: One possibility is $a = 0 \in F$. If $a \neq 0$, then what can you say?

Problem 4 Recall $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ is a linear space over the field $\mathbb{C}$.

(a) Give an example of a subset $S \subset \mathbb{C}^2$ which is closed under scaling, i.e., if $v \in S$, then $\alpha v \in S$ for every $\alpha \in \mathbb{C}$, but S is **not** a subspace of $\mathbb{C}^2$.

(b) Give an example of a subset S of $\mathbb{C}^2$ which is closed under addition, i.e., if $v_1, v_2 \in S$, then $v_1 + v_2 \in S$, but S is **not** a subspace of $\mathbb{C}^2$.

Hint(s): Replace the "big" linear space $\mathbb{C}^2$ with $\mathbb{R}^2$ (a linear space over the field $\mathbb{R}$) where you can draw pictures.

(c) Recall $\mathbb{C}$ is or may be considered a linear space over itself. Give an example of a subset $S \subset \mathbb{C}$ which is closed under scaling, but is **not** a subspace of $\mathbb{C}$.

Problem 5 Assume X_1 and X_2 are subspaces of a linear space X (over a field F). Consider

$$W = \{v_1 + v_2 : v_j \in X_j,\ j = 1, 2\}.$$

(a) Show W is a subspace of X.

(b) Show $W = \operatorname{span}(X_1 \cup X_2)$.

Problem 6 Show that if

$$W = \bigoplus_{n=1}^{k} X_n$$

is the direct sum of subspaces $X_1, X_2, \ldots, X_k$ in a linear space X, then

$$\bigcap_{n=1}^{k} X_j = \{\mathbf{0}\}.$$

Can you strengthen this assertion (conclusion)?

Problem 7 Show the linear space $\mathcal{P}(\mathbb{C})$ of polynomials is not finite dimensional.

Problem 8 If

$$\{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_\ell\}$$

is a basis for a linear space X, then show

$$X = \operatorname{span}\{v_1, v_2, \ldots, v_k\} \bigoplus \operatorname{span}\{w_1, w_2, \ldots, w_\ell\}.$$

Hint: Show that in general if W_1 and W_2 are subspaces of a linear space X with $W_1 \cap W_2 = \{\mathbf{0}\}$, then

$$\operatorname{span}\left(W_1 \bigcup W_2\right) = W_1 \bigoplus W_2.$$

Problem 9 Consider the linear space of polynomials

$$\mathcal{P}_3(\mathbb{C}) = \left\{ p : p(z) = \sum_{n=0}^{3} \alpha_n z^n,\ \ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \right\}$$

of degree no more than 3 over the field $\mathbb{C}$. Find a basis for $\mathcal{P}_3(\mathbb{C})$ consisting of polynomials none of which have degree 2. (Show in detail that the basis you have given is a basis.)

Problem 10 Consider the following proposition:

> If $\{v_1, v_2, v_3, v_4\}$ is a basis for a linear space X, and W is a subspace of X for which
>
> **(a)** $v_1, v_2 \in W$, and
>
> **(b)** $v_3, v_4 \notin W$,
>
> then $\{v_1, v_2\}$ is a basis for W.