Assignment 1: Complex Numbers (a solution)
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Problem 1 Recall the following:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

denotes the set of rational numbers where $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. A **group** is a set G together with an operation, i.e., a rule for combining two elements of G to obtain an element of G, denoted by some particular symbol, for example "$*$," and satisfying the three conditions below. The result of combining two elements g and h in G is denoted by $g * h$.

(i) The operation is associative: $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ whenever $g_1, g_2, g_3 \in G$.

(ii) There exists an identity element, denoted by some particular symbol, for example $e \in G$: $g * e = e * g = g$ whenever $g \in G$.

(iii) Given any $g \in G$, there exists an element $h \in G$ for which $h * g = g * h = e$. This is the existence of inverses, and the element h is called the inverse of g, sometimes denoted g^{-1}.

Consider the set

$$S = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} : (p, q) \neq (0, 0)\}.$$

An operation of "addition" denoted by "$+$" is defined on S by

$$(p, q) + (r, s) = (pr - qs, ps + qr).$$

Show S is a group under this operation.

Problem 2 Given $z \in \mathbb{C} \backslash \{0\}$, use the real arctangent function $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ to express the unique angle θ with $-\pi < \theta \leq \pi$ with

$$\begin{cases} \cos\theta = x/|z| \\ \sin\theta = y/|z| \end{cases}$$

Problem 3 (Boas problem 2.5.25) Show in two different ways

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

for complex numbers $z \in \mathbb{C}$ and $w \in \mathbb{C}\backslash\{0\}$.

(i) Write $z = x + iy$ and $w = \xi + i\eta$. Calculate directly.

(ii) Write $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$. (Show this is possible even when $z = 0$ and assume Euler's exponential formula.) Write also $w = \xi + i\eta = |w|(\cos\phi + i\sin\phi) = |w|e^{i\phi}$. Assume complex exponentiation rules.

Problem 4 (Boas 2.5.62,63)

(a) Draw $\{z \in \mathbb{C} : z^2 = \bar{z}^2\}$.

(b) Draw: $\{z \in \mathbb{C} : |z+1| + |z-1| = 8\}$.

Problem 5 (Boas 2.10.9,10,12)

(a) Write $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\phi + i\sin\phi)$ and show

$$zw = |zw|(\ \cos(\theta+\phi) + i\sin(\theta+\phi)\ ).$$

(b) Find $\sqrt[5]{32}$, i.e., the complex fifth roots of 32.

(c) Find $\sqrt[3]{-1}$, i.e., the complex cube roots of -1.

(d) Find $\sqrt[5]{1}$, i.e., the complex fifth roots of 1.

Problem 6 Assume $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable so that all complex limits

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \tag{1}$$

exist whenever $z \in \mathbb{C}$. Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x,y) = \operatorname{Re} f(x+iy)$. Show the real limit

$$\frac{\partial u}{\partial x}(x,y) = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

exists whenever $(x,y) \in \mathbb{R}^2$ and

$$\operatorname{Re} f'(x+iy) = \frac{\partial u}{\partial x}(x,y).$$

Problem 7 Given a natural number $n \in \mathbb{N}$, consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n$. Use the definition of the complex derivative (1) from Problem 6 involving the limit of a difference quotient to show

$$f'(z) = nz^{n-1}.$$

Problem 8 Assume the series

$$\sum_{n=1}^{\infty} w_n$$

of complex numbers converges to a complex number $\sigma \in \mathbb{C}$. Write down precisely what this means and show the series

$$\sum_{n=1}^{\infty} \operatorname{Im} w_n$$

converges to a real number.

Question: Is this "self evident?"

My answer: "no."

Preliminary thinking: I need to show there is some real number $a \in \mathbb{R}$ for which the the following holds:

Given any $t > 0$, there exists some $M > 0$ for which

$$\left| \sum_{n=1}^{k} \operatorname{Im} w_n - a \right| < t \qquad \text{when} \qquad k > M.$$

Now I could at this point write down what is "given" in the problem, that is, what I am given in order to verify the thing I need to show above. That could also be included in "preliminary thinking," but I note that this is stated as part of the problem, so I'll include that in my solution. which is on the next page. (You may at any time work out the details of your own solution for this problem instead of reading my solution. You can learn that way.)

Solution: I am given that there is a complex number $\sigma \in \mathbb{C}$ for which the following holds:

Given any $\epsilon > 0$, there exists some $N > 0$ for which

$$\left| \sum_{n=1}^{k} w_n - \sigma \right| < \epsilon \qquad \text{when} \qquad k > N.$$

This is precisely what it means to say

$$\sum_{n=1}^{\infty} w_n = \sigma \in \mathbb{C}.$$

Note that one is "instructed" in the problem to "write down precisely what this means."

Let $t > 0$ and let $a = \operatorname{Im} \sigma$. Replacing ϵ with t in the given, I obtain some $N > 0$ so that for $k > N$ there holds

$$\left| \sum_{n=1}^{k} w_n - \sigma \right| < t \qquad \text{when} \qquad k > N.$$

Note that the condition here involves the complex modulus. Furthermore

$$\sum_{n=1}^{k} w_n = \operatorname{Re}\left(\sum_{n=1}^{k} w_n \right) + i \ \operatorname{Im}\left(\sum_{n=1}^{k} w_n \right) = \sum_{n=1}^{k} \operatorname{Re} w_n + i \sum_{n=1}^{k} \operatorname{Im} w_n$$

and

$$\sum_{n=1}^{k} w_n - \sigma = \left(\sum_{n=1}^{k} \operatorname{Re} w_n - \operatorname{Re} \sigma \right) + i \left(\sum_{n=1}^{k} \operatorname{Im} w_n - \operatorname{Im} \sigma \right).$$

This means

$$\left| \sum_{n=1}^{k} w_n - \sigma \right| = \sqrt{\left(\sum_{n=1}^{k} \operatorname{Re} w_n - \operatorname{Re} \sigma \right)^2 + \left(\sum_{n=1}^{k} \operatorname{Im} w_n - \operatorname{Im} \sigma \right)^2},$$

and I know that for $k > N$ there holds

$$\sqrt{\left(\sum_{n=1}^{k} \operatorname{Re} w_n - \operatorname{Re} \sigma \right)^2 + \left(\sum_{n=1}^{k} \operatorname{Im} w_n - \operatorname{Im} \sigma \right)^2} \ < \ t.$$

Now I will use the general inequality $|\operatorname{Im} z| \leq |z|$ where the quantity on the left is a (real) absolute value and the quantity on the right is a complex modulus. Note that this general inequality holds because

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geq \sqrt{(\operatorname{Im} z)^2} = |\operatorname{Im} z|.$$

Here is the crucial estimate: If $M = N$ and $k > M$, then

$$\begin{aligned} \left| \sum_{n=1}^{k} \operatorname{Im} w_n - a \right| &= \left| \sum_{n=1}^{k} \operatorname{Im} w_n - \operatorname{Im} \sigma \right| \\ &\leq \sqrt{\left(\sum_{n=1}^{k} \operatorname{Re} w_n - \operatorname{Re} \sigma \right)^2 + \left(\sum_{n=1}^{k} \operatorname{Im} w_n - \operatorname{Im} \sigma \right)^2} \\ &< t. \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \operatorname{Im} w_n = a = \operatorname{Im} \sigma \in \mathbb{R}.$$

Post solution comment: I have included a fair number of details that seemed reasonable for me to include. One could include more details. One could include fewer details. One can also say the solution is "self evident." All of these considerations involve a fair amount of subjective perspective so to speak. From my perspective, if one can't at least write down the details I've written above, then one can learn something from the process of thinking hard enough to be able to do that. I would not say the solution to the problem is "self evident," but perhaps that is just a matter of taste, i.e., subjective perspective. It just depends on what you think is worth learning.

I will mention as well that there may be errors in my solution above. That doesn't scare me. If you find errors, you can tell me about them, and I will be happy to know they are there so I can correct them. I can learn something that way. Having someone tell me what I have written or expressed is "wrong," is a part of learning. It is not a "penalty." In the case of the solution above particularly, I've typed it up pretty quickly, and I've only proofread it a couple times, so there could easily be errors and/or typos.

Problem 9 Show that if $\{w_n\}_{n=1}^{\infty}$ is a **sequence** of complex numbers with

$$\lim_{n\to\infty} w_n = z_0 \in \mathbb{C},$$

then the sequence $\{w_n\}_{n=1}^{\infty}$ is **bounded**, that is, there exists a fixed positive constant $M > 0$ (independent of n) for which

$$|w_n| < M \qquad \text{for every } n = 1, 2, 3, \ldots.$$

Problem 10 Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial function with values

$$p(z) = \sum_{n=0}^{k} a_n z^n$$

with $k \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_k \in \mathbb{C}$ and $a_k \neq 0$. Note that if $p(z) \neq 0$ for every $z \in \mathbb{C}$, then the function $q : \mathbb{C} \to \mathbb{C}$ by

$$q(z) = \frac{1}{p(z)}$$

is well-defined. Assume p has no zeros, and show q is complex differentiable on $\mathbb{C}$ and **bounded above**, that is, there is some fixed positive constant $M > 0$ (independent of z) for which

$$|q(z)| < M \qquad \text{for every } z \in \mathbb{C}.$$

One says q is a bounded entire function.