Assignment 1: Complex Numbers
Due Thursday September 10, 2026

John McCuan

Problem 1 Recall the following:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

denotes the set of rational numbers where $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. A **group** is a set G together with an operation, i.e., a rule for combining two elements of G to obtain an element of G, denoted by some particular symbol, for example "$*$," and satisfying the three conditions below. The result of combining two elements g and h in G is denoted by $g * h$.

(i) The operation is associative: $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ whenever $g_1, g_2, g_3 \in G$.

(ii) There exists an identity element, denoted by some particular symbol, for example $e \in G$: $g * e = e * g = g$ whenever $g \in G$.

(iii) Given any $g \in G$, there exists an element $h \in G$ for which $h * g = g * h = e$. This is the existence of inverses, and the element h is called the inverse of g, sometimes denoted g^{-1}.

Consider the set

$$S = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} : (p, q) \neq (0, 0)\}.$$

An operation of "addition" denoted by "$+$" is defined on S by

$$(p, q) + (r, s) = (pr - qs, ps + qr).$$

Show S is a group under this operation.

Problem 2 Given $z \in \mathbb{C}\backslash\{0\}$, use the real arctangent function $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ to express the unique angle θ with $-\pi < \theta \leq \pi$ with

$$\begin{cases} \cos\theta = x/|z| \\ \sin\theta = y/|z| \end{cases}$$

Problem 3 (Boas problem 2.5.25) Show in two different ways

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

for complex numbers $z \in \mathbb{C}$ and $w \in \mathbb{C}\backslash\{0\}$.

(i) Write $z = x + iy$ and $w = \xi + i\eta$. Calculate directly.

(ii) Write $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$. (Show this is possible even when $z = 0$ and assume Euler's exponential formula.) Write also $w = \xi + i\eta = |w|(\cos\phi + i\sin\phi) = |w|e^{i\phi}$. Assume complex exponentiation rules.

Problem 4 (Boas 2.5.62,63)

(a) Draw $\{z \in \mathbb{C} : z^2 = \bar{z}^2\}$.

(b) Draw: $\{z \in \mathbb{C} : |z+1| + |z-1| = 8\}$.

Problem 5 (Boas 2.10.9,10,12)

(a) Write $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\phi + i\sin\phi)$ and show

$$zw = |zw|(\ \cos(\theta+\phi) + i\sin(\theta+\phi)\ ).$$

(b) Find $\sqrt[5]{32}$, i.e., the complex fifth roots of 32.

(c) Find $\sqrt[3]{-1}$, i.e., the complex cube roots of -1.

(d) Find $\sqrt[5]{1}$, i.e., the complex fifth roots of 1.

Problem 6 Assume $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable so that all complex limits

$$f'(z) = \lim_{w\to z} \frac{f(w) - f(z)}{w - z} \tag{1}$$

exist whenever $z \in \mathbb{C}$. Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x,y) = \operatorname{Re} f(x+iy)$. Show the real limit

$$\frac{\partial u}{\partial x}(x,y) = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

exists whenever $(x,y) \in \mathbb{R}^2$ and

$$\operatorname{Re} f'(x+iy) = \frac{\partial u}{\partial x}(x,y).$$

Problem 7 Given a natural number $n \in \mathbb{N}$, consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n$. Use the definition of the complex derivative (1) from Problem 6 involving the limit of a difference quotient to show

$$f'(z) = nz^{n-1}.$$

Problem 8 Assume the series

$$\sum_{n=1}^{\infty} w_n$$

of complex numbers converges to a complex number $\sigma \in \mathbb{C}$. Write down precisely what this means and show the series

$$\sum_{n=1}^{\infty} \operatorname{Im} w_n$$

converges to a real number.

Problem 9 Show that if $\{w_n\}_{n=1}^{\infty}$ is a **sequence** of complex numbers with

$$\lim_{n \to \infty} w_n = z_0 \in \mathbb{C},$$

then the sequence $\{w_n\}_{n=1}^{\infty}$ is **bounded**, that is, there exists a fixed positive constant $M > 0$ (independent of n) for which

$$|w_n| < M \qquad \text{for every } n = 1, 2, 3, \ldots.$$

Problem 10 Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial function with values

$$p(z) = \sum_{n=0}^{k} a_n z^n$$

with $k \in \mathbb{N}_0$ and $a_0, a_1, \ldots, a_k \in \mathbb{C}$ and $a_k \neq 0$. Note that if $p(z) \neq 0$ for every $z \in \mathbb{C}$, then the function $q : \mathbb{C} \to \mathbb{C}$ by

$$q(z) = \frac{1}{p(z)}$$

is well-defined. Assume p has no zeros, and show q is complex differentiable on $\mathbb{C}$ and **bounded above**, that is, there is some fixed positive constant $M > 0$ (independent of z) for which

$$|q(z)| < M \qquad \text{for every } z \in \mathbb{C}.$$

One says q is a bounded entire function.