The equality of ratios of areas in one given figure to the ratios of corresponding areas in a similar figure follows as before:

$$\frac{A_1'}{A_2'} = \frac{A_1}{A_2}.$$

Applying this principle concerning ratios of areas to the right triangle on the right in Figure 1.6 we have

$$\frac{a^2}{A_1} = \frac{c^2}{A_1 + A_1'} = \frac{b^2}{A_1'}.$$

Therefore,

$$a^2 = \frac{c^2 A_1}{A_1 + A_1'} \qquad \text{and} \qquad b^2 = \frac{c^2 A_1'}{A_1 + A_1'}.$$

Thus,

$$a^2 + b^2 = c^2 \left(\frac{A_1}{A_1 + A_1'} + \frac{A_1'}{A_1 + A_1'} \right) = c^2. \qquad \square$$

Exercise 1.1 (pre-Cartesian Euclidean length) Show the angle bisector at A in a triangle with vertices A, B, and C intersects the side opposite vertex A in a point P for which

$$\frac{\text{length}\left(\overline{PB}\right)}{\text{length}\left(\overline{AB}\right)} = \frac{\text{length}\left(\overline{PC}\right)}{\text{length}\left(\overline{AC}\right)}.$$

1.3 Descartes: precalculus

Let me begin this section with a reformulation of Euclid's definition of a point in terms that emphasize both the distinction suggested by Spengler and the "developments" I wish to present below:

> A **point** is an indivisible body by itself.

Spengler argues that Fermat was primarily representative of the abandoning of the classical mind and the introduction of an entirely different western cultural tradition. At least Spengler presents Fermat's perception of a relation like

$$x^2 + y^2 = z^2$$

as truly indeterminate (or containing variables) as opposed simply to a symbolic representation of specific examples $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$ presented in classical "bodily form" so to speak. Spengler presents the transition as a somewhat enigmatic and novel one, but one that was at length unavoidable. For Euclid[3] any meaning beyond the "bodily presentation" was unthinkable. Putting the inovation of Descartes in contrast to Euclid's definition of a point provides in our context an even better example. Here might be the updated version of Euclid's definition which to us is very familiar:

> A **point** is a location in a Euclidean space with coordinates.

From this perspective the "spaces" $\mathbb{R}^n$ are the base and ground for all geometric consideration. In principle, this is obviously limiting. Computationally, however, Descartes' point of view may be viewed as rather fruitful if not revolutionary. It is also very familiar, and it survives in some form in the concept of a Riemannian manifold the understanding of which is, in a certain sense, our ultimate objective.

This is our first mention of $\mathbb{R}^n$ which plays a key role in all that follows, so we take a moment to give something of a formal introduction. Instead of the bodily point of Euclid, we now have many points considered together, and to consider them, it is natural to first pick a dimension. Location, and hence point, is from this perspective connected to number. Associated with dimension $n = 0$ we can take the single number 0. Thus, in dimension zero there is a single solitary point with a single solitary location. It is not the presence of something at $x = 0$, but rather the absence represented by the point. Paradoxically or as Spengler would say with inherent contradiction, in this point of view a point itself is the presence of absence.

For $n = 1$, we obtain the foundational case of the real numbers $\mathbb{R}$. We can distinguish the complete Archimedian field from the real vector space or the Riemannian manifold of dimension one by using $\mathbb{R}^1$, though we will not be very strict about this. From there, we have

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

Many of the properties of the field $\mathbb{R}$ and the vector spaces $\mathbb{R}^n$ should be familiar, though we will discuss some of them explicitly below especially when

[3]. . . or any "classical man" as Spengler styles humans sharing that particular culture which he perceives as well-defined

the properties in question are important for the discussion of Riemannian manifolds.

We have mentioned Descartes whose name is associated with the Cartesian coordinates of what we call the Euclidean spaces $\mathbb{R}^n$, and we have mentioned Fermat though we have not mentioned Diophantus (c. 210–290 AD) who Spengler describes as anticipating the non-classical western culture (or equivalently mathematics) involving space and equations with variables but lacking the language to express that mind. There is a nice story to tell about Descartes and Fermat, and the story is relevant to the direction of our presentation.

Folium of Descartes

Desartes challenged Fermat to find the equation of the tangent line at each point to the curve $\{(x, y) \in \mathbb{R}^2 : x^3 + y^3 = 3axy\}$ now known as the folium of Descartes.

Exercise 1.2 (folium of Descartes) There are many approaches to giving a solution to the problem with which Descartes challenged Fermat now that calculus is available to us. The steps **(a)**-**(d)** given below in this exercise outline one such approach. Preliminary to consideration of **(a)**-**(d)**, you might challenge yourself:

(i) Give a solution of Descartes' challenge problem without the use of calculus as Fermat was able to do.

(ii) Give your own solution of Descartes' challenge problem.

Here is an outline of a solution:

(a) Consider alternative coordinates (ξ, η) obtained by rotating the plane by an angle $\pi/4$. Specifically, if

$$\mathbf{p} \in \Gamma = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 = 3axy\},$$

then the new coordinates of $\mathbf{p}$ are given by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos[\pi/4] & \sin[\pi/4] \\ -\sin[\pi/4] & \cos[\pi/4] \end{pmatrix} \mathbf{p} \tag{1.3}$$

so that the folium in the new coordinates is given by

$$\left\{ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos[\pi/4] & \sin[\pi/4] \\ -\sin[\pi/4] & \cos[\pi/4] \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : x^3 + y^3 = 3axy \right\}.$$

That is,

$$\left(\frac{\xi-\eta}{\sqrt{2}}\right)^3 + \left(\frac{\xi+\eta}{\sqrt{2}}\right)^3 = 3a\left(\frac{\xi-\eta}{\sqrt{2}}\right)\left(\frac{\xi+\eta}{\sqrt{2}}\right). \tag{1.4}$$

The expressions in (1.3) and (1.4) may themselves be obtained in various ways. One way is to note that the coordinates (ξ, η) of the point $\mathbf{p}$ on the left in Figure 1.7 are obtained by a clockwise rotation of the plane by $\pi/4$ so that (1.3) holds.

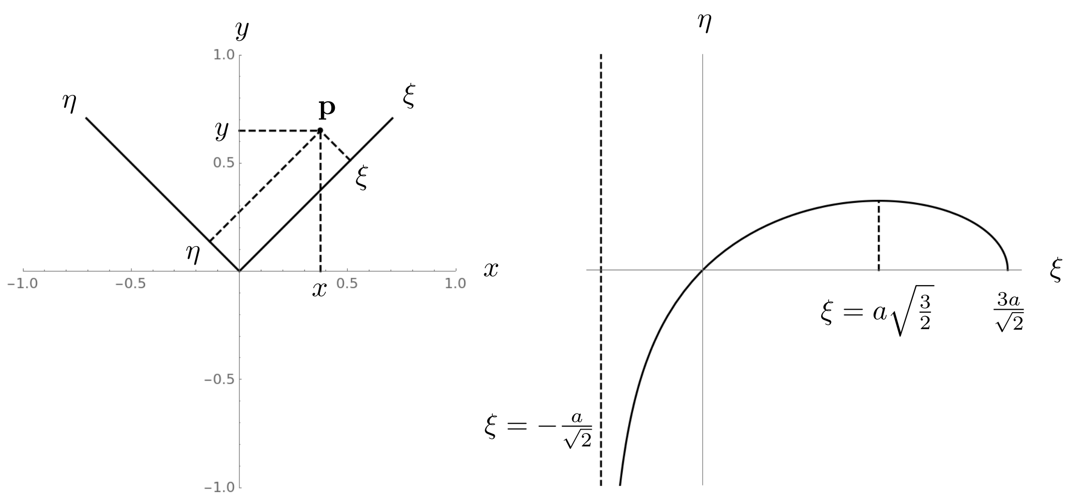

Figure 1.7: Rotation of coordinates and the folium of Descartes

(b) For $a > 0$, plot the affine functions $g_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, 2$ by $g_1(\xi) = 3a - \xi\sqrt{2}$ and $g_2(\xi) = \xi\sqrt{2} + a$ in the same ξ, η-plane to find the common interval where the quotient

$$\frac{g_1(\xi)}{g_2(\xi)}$$

is positive.

(c) Using what you found in part **(b)** solve (1.4) for η as a function of ξ on an appropriate interval to obtain a function $f : (-a/\sqrt{2}, 3a/\sqrt{2}] \to \mathbb{R}$ by

$$f(\xi) = \frac{\xi}{\sqrt{3}} \sqrt{\frac{3a - \xi\sqrt{2}}{\xi\sqrt{2} + a}}$$

for which the folium is given by

$$\Gamma = \left\{ (\xi, \pm f(\xi)) : -\frac{a}{\sqrt{2}} < \xi \le \frac{3a}{\sqrt{2}} \right\}.$$

(d) Verify that $f(0) = 0 = f(3a/\sqrt{2})$, and use calculus to show the function f in part **(c)** increases to a unique maximum at $\xi = a\sqrt{3/2}$ and satisfies

$$\lim_{\xi \searrow -\frac{a}{\sqrt{2}}} f(\xi) = \lim_{\xi \nearrow \frac{3a}{\sqrt{2}}} f'(\xi) = -\infty$$

so the graph looks like the one illustrated on the right in Figure 1.7.

(e) Find the equation for the tangent line to each point of the folium in the rotated ξ, η-coordinates.

(f) Rotate back to find the equation for the tangent line at each point in the original x, y-coordinates.

(g) Consider the cases $a = 0$ and $a < 0$.

(h) Use mamthematical software to plot the folium in the original x, y-coordinates and verify the equations of the tangent lines at the origin.

Before turning to the consideration of calculus in more general terms in relation with geometry, I will attempt to cast the three problems of Euclidean geometry into this new framework.

Descartes' length problem (Problem 1): Given two points

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \qquad \text{and} \qquad \mathbf{y} = (y_1, y_2, \dots, y_n)$$

in $\mathbb{R}^n$ find the length of the line segment connecting $\mathbf{x}$ to $\mathbf{y}$ in terms of in terms of the Cartesian coordinates of $\mathbf{x}$ and $\mathbf{y}$.

Descartes' angle problem (Problem 2): Given two points

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \qquad \text{and} \qquad \mathbf{y} = (y_1, y_2, \dots, y_n)$$

in $\mathbb{R}^n$ find the angle between of the line segments connecting $\mathbf{0}$ to $\mathbf{x}$ and $\mathbf{0}$ to $\mathbf{y}$ in terms of in terms of the Cartesian coordinates of $\mathbf{x}$ and $\mathbf{y}$.

Descartes' area problem (Problem 3): Given a triangle with vertices $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$, and $\mathbf{z} = (z_1, z_2, \dots, z_n)$ in $\mathbb{R}^n$, find the area of the triangle in terms of the Cartesian coordinates of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

More extensive comments on each of these essentially important problems may be found in various places below. Perhaps I will only mention at this point something about the answers. The first problem leads to a generalization

$$\ell = \sqrt{\sum_{j=1}^{n} (y_j - x_j)^2} \tag{1.5}$$

of the formula (1.1). The angle problem leads to consideration of the quantity

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j \tag{1.6}$$

which it will be noted enjoys an interesting relation to the quantity in (1.5).

Exercise 1.3 Express the quantity ℓ in (1.5) as an appropriate expression in terms of the dot product appearing in (1.6).

It is not entirely clear if (1.5) is properly a solution to the problem, that is a calculation giving the solution, or simply the definition of the length. Similarly, for the angle problem one either calculates the angle to be or defines it as

$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|\ |\mathbf{y}|} \right) \tag{1.7}$$

where $|\mathbf{x}|$ and $|\mathbf{y}|$ denote the lengths of the segments from $\mathbf{0}$ to $\mathbf{x}$ and from $\mathbf{0}$ to $\mathbf{y}$ respectively.

Overall, the topic of **linearity** and the **vector space structure of** $\mathbb{R}^n$ comes to one's attention in these problems, and these are both important and considered in more detail below.

As with the Euclidean area problem there are various approaches to the Cartesian area problem posed above. By use of linearity, the points $\mathbf{x}$ and $\mathbf{y}$ may be taken as arbitrary and the third point $\mathbf{z}$ may be taken to be $\mathbf{0} \in \mathbb{R}^n$ in the third problem. After this reduction, the calculation/definition gives

$$\frac{1}{2}\ |\mathbf{x}|\ |\mathbf{y}|\ \sin\theta = \frac{1}{2}\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2}. \tag{1.8}$$

Linearity again suggests the consideration of the parallelogram

$$\{a\mathbf{x} + b\mathbf{y} : a, b \in [0,1]\} \tag{1.9}$$

with area twice that of the triangle or

$$\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2}. \tag{1.10}$$

Expanding the quantity inside the square root, one finds

$$\begin{aligned}
|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2 &= \left(\sum_{j=1}^n x_j^2\right)\left(\sum_{j=1}^n y_j^2\right) - \left(\sum_{j=1}^n x_j y_j\right)^2 \\
&= \sum_{i=1}^n \left[\sum_{j=1}^n x_i^2 y_j^2 - \sum_{j=1}^n x_i y_i x_j y_j\right] \\
&= \sum_{i=1}^n \left[\sum_{j=1, j\neq i}^n x_i^2 y_j^2 - \sum_{j=1, j\neq i}^n x_i y_i x_j y_j\right].
\end{aligned}$$

This is an interesting quantity. Specializing to the case $n = 2$ for example, we have

$$x_1^2 y_2^2 - x_1 y_1 x_2 y_2 + x_2^2 y_1^2 - x_2 y_2 x_1 y_1 = (x_1 y_2 - x_2 y_1)^2,$$

so the area of the parallelogram becomes

$$\sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2} = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right|. \tag{1.11}$$

Exercise 1.4 Show the square of the area of the parallelogram spanned by $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfies

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2 = \det(A^T A)$$

where A is the $n \times 2$ matrix

$$A = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}$$

with $\mathbf{x}$ and $\mathbf{y}$, or more properly $\mathbf{x}^T$ and $\mathbf{y}^T$ in the columns.

Exercise 1.5 The linear translation preceeding the formula (1.8) if executed properly and in detail leads to replacing $\mathbf{x} - \mathbf{z}$ with $\mathbf{x}$, $\mathbf{y} - \mathbf{z}$ with $\mathbf{y}$ and $\mathbf{z} - \mathbf{z} = \mathbf{0}$ with $\mathbf{0}$. One of these replacements is notationally justified. Repeat the discussion above starting with (1.8) but use instead the **vectors** $\mathbf{v} = (v_1, v_2, \ldots, v_n) = \mathbf{x} - \mathbf{z}$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n) = \mathbf{y} - \mathbf{z}$. In particular, obtain alternative formulas for (1.11) and in Exercise 1.4 in terms of (the original) $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Exercise 1.6 Based on your solution to Exercise 1.5 above, you should have a formula for the parallelogram spanned by vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ from (1.9). Based on the observation that the line segment from $\mathbf{x}$ to $\mathbf{y}$ is parameterized in Cartesian coordinates by $(1-t)\mathbf{x} + t\mathbf{y}$, show the parallelogram consists of precisely the line segments connecting points in the edge segments

$$\begin{aligned}
\Gamma_1 &= \{a\mathbf{v} : 0 \le a \le 1\}, \\
\Gamma_2 &= \{b\mathbf{w} : 0 \le b \le 1\}, \\
\Gamma_3 &= \{\mathbf{v} + b\mathbf{w} : 0 \le b \le 1\}, \text{ and} \\
\Gamma_4 &= \{\mathbf{w} + a\mathbf{v} : 0 \le a \le 1\}.
\end{aligned}$$

Draw pictures to illustrate the edges and the six different kinds of internal segments you considered.

1.4 Newton: general paths and areas

Naturally we can associate the topics of differentiation and integration with Isaac Newton, though an association with Gottfried Wilhelm Leibniz (1646–1716) would be equally natural. Here the framework of Cartesian coordinates is assumed and one encounters naturally the **graph** of a function and the spaces of functions $C^0[a,b]$ and $C^1[a,b]$ consisting of real valued functions defined on a closed interval $[a,b]$ of the real line with $a < b$ which are continuous and continuously differentiable respectively.

To emphasize the change of perspective involved in the revolutionary ideas of calculus, I immediately start by stating versions of the three geometric problems given (twice) above:

Newton's length problem (Problem 1): Given two points (a, y_a) and (b, y_b) in $\mathbb{R}^2$ with $a, b \in \mathbb{R}$ satisfying $a < b$, find the length of the graph

$$\{(x, h(x)) : a \le x \le b\}$$

of a continuously differentiable function $h : [a,b] \to \mathbb{R}$ in

$$\{f \in C^1[a,b] : f(a) = y_a \text{ and } f(b) = y_b\}$$

as indicated in Figure 1.8.

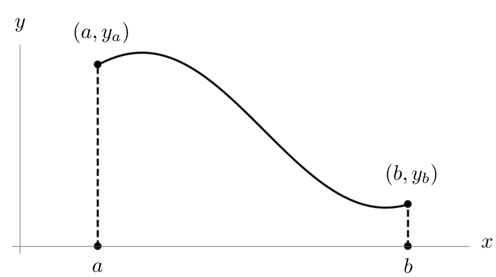

Figure 1.8: A C^1 graph connecting two points in the plane.

Newton's angle problem (Problem 2): Given two C^1 paths $\alpha : [a_1, b_1] \to \mathbb{R}^n$ and $\beta : [a_2, b_2] \to \mathbb{R}^n$ where $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$ for $j = 1, 2$ satisfying

$$\alpha(t_1) = \beta(t_2)$$

for some $t_j \in (a_j, b_j)$ for $j = 1, 2$, find the angle at which the paths meet at the common point $\alpha(t_1) = \beta(t_2)$.

Newton's area problem (Problem 3): Given $a, b \in \mathbb{R}$ with $a < b$, find the area of the region

$$\{(x, y) : 0 \leq y \leq h(x),\ a \leq x \leq b\}$$

determined by a nonnegative continuous function $h : [a, b] \to \mathbb{R}$. Such a region is indicated in Figure 1.9.

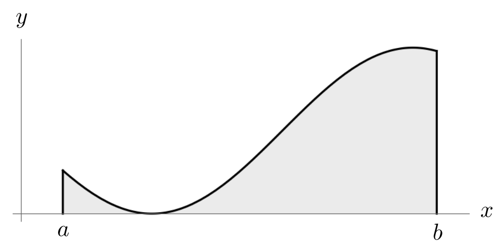

Figure 1.9: A C^0 graph enclosing (along with the horizontal axis and two vertical segments) an area in the plane.

You have likely noticed that a precipitous drop in dimension has occurred in Newton's length problem and Newton's area problem relative to the corresponding problems associated with Descartes above. There are reasons for this, but there are also generalizations to higher dimensions.

Exercise 1.7 Solve Newton's length problem as stated above using **polygonal approximation** as follows:

(a) Based on a **partition** $\mathcal{P} = \{x_0 = a < x_1 < x_2 < \cdots < x_k = b\}$ of the interval find the sum of the lengths $|(x_{j+1}, h(x_{j+1})) - (x_j, h(x_j))|$ of the line segments connecting consecutive pairs of points.

(b) Take the **limit** of this sum as the norm of the partition

$$\|\mathcal{P}\| = \max\{x_{j+1} - x_j : j = 0, 1, 2, \ldots, k - 1\}$$

tends to zero.

Exercise 1.8 State and solve a generalization of Newton's length problem for parameterized curves $\alpha : [a, b] \to \mathbb{R}^n$.

Exercise 1.9 Explain how the solution of Exercise 1.8 applies to Exercise 1.7.

Exercise 1.10 Solve Newton's angle problem. Be careful, it may be that Newton was careless, and you may need to include additional hypotheses.

Exercise 1.11 Specialize Newton's angle problem to graphs of C^1 functions of one variable.

Exercise 1.12 Consider Newton's area problem in the special case of a parallelogram spanned by $a\mathbf{e}_1 = (a, 0) \in \mathbb{R}^2$ and $\mathbf{w} = (w_1, w_2)$ with $a, w_j > 0$ for $j = 1, 2$. (Using Newton's approach, do you get the same answer obtained in Exercise 1.5)?

Exercise 1.13 Consider Newton's area problem in the special case of a parallelogram spanned by $a\mathbf{e}_1 = (a, 0) \in \mathbb{R}^2$ and $b\mathbf{e}_2 = (0, b)$ with $a, b > 0$ for $j = 1, 2$. (Using Newton's approach, do you get the same answer obtained in Exercise 1.5?

Exercise 1.14 Consider Newton's area problem in the special case of a parallelogram spanned by $a\mathbf{e}_1 = (a, 0) \in \mathbb{R}^2$ and $\mathbf{w} = (w_1, w_2)$ with $w_1 < 0 < a, w_2$. (Using Newton's approach, do you get the same answer obtained in Exercise 1.5)?

Exercise 1.15 Show that given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ with $|\mathbf{v}| = a > 0$, there is a unique rotation $\mathbf{p} : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane for which $\mathbf{p}(\mathbf{v}) = a\mathbf{e}_1$. What happens if you try to use this approach in an effort to apply Exercises 1.12-1.14 to obtain the assertions of Exercise 1.5?

Exercise 1.16 What happens if you try to generalize Exercise 1.15 to the case of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ for $n > 2$?

Exercise 1.17 Consider the following generalization of Newton's area problem: Given $a, b \in \mathbb{R}$ and $\epsilon > 0$, by a **cyclic path** $\alpha \in C^1([a - \epsilon, b + \epsilon] \to \mathbb{R}^2)$ we mean a function each of whose values is given by a pair $\alpha(t) = (\alpha_1(t), \alpha_2(t)$ where $\alpha_j \in C^1[a - \epsilon, b + \epsilon]$ is a continously differentiable real valued function for $j = 1, 2$ and for which we require

$$\alpha(t_1) \neq \alpha(t_2) \qquad \text{for} \qquad a \leq t_1 < t_2 < b,$$

$$\alpha(t) = \alpha(b - a + t) \qquad \text{for} \qquad a - \epsilon \leq t \leq a,$$

$$\alpha(t) = \alpha(a + t - b) \qquad \text{for} \qquad b \leq t \leq b + \epsilon,$$

and $|\alpha'(t)| \neq 0$, what is the area of the region A enclosed by

$$\Gamma = \{\alpha(t) : t \in [a, b]\}?$$

(a) In what way is this **not** a generalization of Newton's area problem?

(b) Prove $\mathbb{R}^2 \backslash \Gamma = A \cup B$ where A and U are connected, disjoint open sets with U unbounded.[4]

(c) Newton's answer[5] to this problem might be along the following lines: Let $\mathcal{P} = \{A_j \cap A\}_{j=1}^k$ be a collection of (closed) rectangles (intersected with the region A). We say $\mathcal{P}$ is a **partition** of the region A if the rectangles have intersections $A_i \cap A_j$ contained in the union of the edges of the rectangles when $i \neq j$ and

$$A = \bigcup_{j=1}^{k} (A_j \cap A).$$

Define the **norm of the partition** by

$$\|\mathcal{P}\| = \max\{\operatorname{diam}(A_j) : j = 1, 2, \ldots, k\}$$

where

$$\operatorname{diam}(S) = \sup\{|\mathbf{y} - \mathbf{x}| : \mathbf{x}, \mathbf{y} \in S\}.$$

Then

$$\operatorname{area}(A) = \int_A 1 = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{k} \operatorname{area}(A_j \cap A).$$

[4] If you do not know enough topology to make sense of all the terminology in this problem, you can look ahead to Chapter 13 and read about topics like topological spaces, open sets, and connected sets. This is a simple version of the famous Jordan curve theorem.

[5] Technically, this might be more properly recognized as Riemann's answer, but we're making no pretense to historical accuracy here. In fact, you may wish to look ahead at Chapter 3 to see more of what Riemann had in mind.

(i) It might be objected that finding the area of $A_j \cap A$ is just as difficult as finding the area of the region A, at least when $A_j \backslash A \neq \phi$. Show that even though the value of area$(A_j \cap A)$ may be difficult to compute one has

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{j=1, A_j \backslash A \neq \phi}^{k} \text{area}(A_j \cap A) = 0$$

so that

$$\text{area}(A) = \int_A 1 = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1, A_j \subset A}^{k} \text{area}(A_j).$$

(ii) Show there exist partitions by rectangles of the region A as described above.

(iii) Find a sequence of partitions $\mathcal{P}_m$ of the region A by rectangles with

$$\lim_{m \to \infty} \|\mathcal{P}_m\| = 0.$$

(d) Gauss' answer might be

$$\text{area}(A) = \frac{1}{2} \left| \int_{\mathbf{x} \in \Gamma} \mathbf{x} \cdot \mathbf{n} \right| \tag{1.12}$$

where

$$\mathbf{n} = \mathbf{n}(\mathbf{x}) = \frac{(-\alpha_2'(t), \alpha_1'(t))}{|\alpha'(t)|}$$

determined by $t \in [a, b]$ with $\alpha(t) = \mathbf{x}$.

(i) Give a definition of the integral appearing in (1.12)

(ii) Justify Gauss' answer.

(e) George Green (1793–1841) might have given a slightly different answer:

$$\text{area}(A) = \frac{1}{2} \left| \int_{\mathbf{x} \in \Gamma} \mathbf{x} \cdot \frac{\alpha'}{|\alpha'|} \right| \tag{1.13}$$

where $\alpha' = \alpha'(t)$ is evaluated in the integral at a value of the parameter $t \in [a, b]$ determined by the condition $\alpha(t) = \mathbf{x}$.

(i) Give a definition of the integral appearing in (1.13)

(ii) Justify Green's answer.

As might be imagined, the revolution represented by calculus motivated many fundamentally new kinds of problems and new kinds of geometry problems in particular. The next three subsections distinguish some of those new problems:

1.4.1 New problem(s): The geometry of functions

We have mentioned the collections of functions $C^0[a, b]$ and $C^1[a, b]$ consisting of certain real valued functions $h : [a, b] \to \mathbb{R}$ defined on a closed interval

$$[a, b] = \{x \in \mathbb{R} : a < x < b\}$$

with $a, b \in \mathbb{R}$. This may be a good time to review and generalize these spaces of functions which are considered in much more detail in Chapter 13. A function $h : [a, b] \to \mathbb{R}$ is **continuous at** $x_0 \in [a, b]$ if for each $\epsilon > 0$, there is some $\delta > 0$ for which

$$|h(x) - h(x_0)| < \epsilon \qquad \text{whenever} \qquad x \in [a, b] \quad \text{and} \quad |x - x_0| < \delta. \quad (1.14)$$

A function $h : [a, b] \to \mathbb{R}$ is **continuous on all of** $[a, b]$ and we write $h \in C^0[a, b]$ if h is continuous at each point $x_0 \in [a, b]$. For such functions it is natural to distinguish a new kind of geometric question:

Newton's first integration problem (Problem 5): Given $h \in C^0[a, b]$, find the integral

$$\int_a^b h(x)\, dx$$

of h.

This problem easily generalizes to higher dimensions and from this point allows a fundamental distinction between the geometry of sets in the domain $A \subset \mathbb{R}^n$ of a real valued function $h : A \to \mathbb{R}$ and the geometry associated with the graph of the function h or with the function h more generally.

In order to make the generalization carefully, we proceed in several steps:

1. If U is an **open** and bounded set in $\mathbb{R}^n$, by which we mean for each $P \in U$, there is some $r > 0$ for which
$$B_r(P) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - P| < r\} \subset U$$
and there is some $R > 0$ for which $U \subset B_R(P)$, we can define the **closure** $\overline{U}$ of U to be the set
$$\overline{U} = \{\mathbf{x} \in \mathbb{R}^n : B_r(\mathbf{x}) \cap U \neq \phi \text{ for every } r > 0\}.$$
This is a/the higher dimensional version of the closed interval $[a, b]$.

2. A function $h : \overline{U} \to \mathbb{R}$ is **continuous at** $P \in \overline{U}$ if for each $\epsilon > 0$, there is some $\delta > 0$ for which
$$|h(\mathbf{x}) - h(P)| < \epsilon \qquad \text{whenever} \qquad \mathbf{x} \in \overline{U} \quad \text{and} \quad |\mathbf{x} - P| < \delta. \tag{1.15}$$

3. A function $h : \overline{U} \to \mathbb{R}$ is **continuous on all of** $\overline{U}$ and we write $h \in C^0(\overline{U})$ if h is continuous at each point $P \in \overline{U}$.

4. The set $\overline{U} \backslash U$ is called the **boundary** of U, and is denoted by ∂U. The boundary of an open bounded subset of $\mathbb{R}^n$ can be a complicated set, so we would like to focus on open and bounded sets U for which the complication of the boundary does not cause us undue consternation in regard to integration. One way to deal with this is the following:

   (a) A **closed cubical cell** in $\mathbb{R}^n$ is a set of the form
   $$\prod_{j=1}^{n} [a_j, b_j] = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ a_j \le x_j \le b_j \text{ for } j = 1, 2, \ldots, n\}$$
   where $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$ for $j = 1, 2, \ldots, n$.

   (b) The **diameter** of a closed cubical cell
   $$C = \prod_{j=1}^{n} [a_j, b_j]$$
   is given by
   $$\operatorname{diam}(C) = \max\{|\mathbf{y} - \mathbf{x}| : \mathbf{x}, \mathbf{y} \in C\}.$$

(c) The n-**dimensional measure** of a closed cubical cell

$$C = \prod_{j=1}^{n} [a_j, b_j]$$

is given by

$$\mathcal{L}(C) = \prod_{j=1}^{n} (b_j - a_j).$$

(d) The **edge-faces** of a closed cubical cell

$$C = \prod_{j=1}^{n} [a_j, b_j]$$

are the sets

$$\begin{aligned} E_\ell = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ a_j \leq x_j \leq b_j \text{ for } j = 1, 2, \ldots, n \text{ and } x_\ell = a_\ell\} \quad \text{and} \\ F_\ell = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ a_j \leq x_j \leq b_j \text{ for } j = 1, 2, \ldots, n \text{ and } x_\ell = b_\ell\} \end{aligned}$$

for $\ell = 1, 2, \ldots, n$. We then set

$$\partial C = \cup_{\ell=1}^{n} (E_\ell \cup F_\ell).$$

The set ∂C is called the **boundary** of the cubical cell C.

(e) A collection $\mathcal{P} = \{C_j\}_{j=1}^{k}$ of closed cubical cells is said to be a **partition** of

$$\bigcup_{j=1}^{k} C_j$$

if $C_i \cap C_j \subset \partial C_i$ whenever $i \neq j$.

(f) The **norm** of a partition $\mathcal{P} = \{C_j\}_{j=1}^{k}$ by cubical cells is given by

$$\|\mathcal{P}\| = \max\{\operatorname{diam}(C_j) : j = 1, 2, \ldots, k\}.$$

(g) The boundary ∂U of a bounded open set $U \subset \mathbb{R}^n$ is said to have **measure zero** if the following condition holds:

$$\lim_{\|\mathcal{P}\|\to 0} \sup \left\{ \sum_{j=1, \partial U \cap C_j \neq \phi}^{k} \mathcal{L}(C_j) : \partial U \subset \bigcup_{j=1}^{n} C_j \right\} = 0$$

meaning for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$\left| \sum_{j=1, \partial U \cap C_j \neq \phi}^{k} \mathcal{L}(C_j) \right| < \epsilon$$

whenever $\mathcal{P} = \{C_j\}_{j=1}^{k}$ is a partition by cubical cells with

$$\partial U \subset \bigcup_{j=1}^{n} C_j.$$

In this case we write $\mathcal{L}(\partial U) = 0$.

Thus, we arrive at the following:

Newton's general integration problem (Problem 5): Given an open bounded set $U \subset \mathbb{R}^n$ with $\mathcal{L}(\partial U) = 0$ and a function $h \in C^0(\overline{U})$, find the integral

$$\int_{\overline{U}} h$$

of h.

In practice, the condition $\mathcal{L}(\partial U) = 0$ in Newton's integration problem is often ensured by some other (stronger) condition(s) on ∂U.

Exercise 1.18 Given a cyclic path $\Gamma \subset \mathbb{R}^2$ of the kind described in Exercise 1.17 above, show $\mathcal{L}(\partial A) = 0$ where A is the bounded open component of $\mathbb{R}^2 \backslash \Gamma$.

Exercise 1.19 Given $h \in C^0(\overline{U})$ with U an open bounded subset of $\mathbb{R}^n$ with $\mathcal{L}(\partial U) = 0$, justify the assertion

$$\int_{\overline{U}} h = \int_{U} h.$$

The main point is that given an open bounded set $U \subset \mathbb{R}^n$ (with $\mathcal{L}(\partial U) = 0$) general integration may be used to understand real valued functions $h \in C^0(\overline{U})$. For example, the "size" of a function $h \in C^0(\overline{U})$ may be considered with respect to the following interesting quantities:

$$\|h\|_{L^1} = \int_U |h|, \tag{1.16}$$

$$\|h\|_{L^2} = \left(\int_U |h|^2\right)^{1/2}, \text{ and} \tag{1.17}$$

$$\|h\|_{L^p} = \left(\int_U |h|^p\right)^{1/p} \quad \text{for} \quad p \geq 1. \tag{1.18}$$

Exercise 1.20 Given an open bounded set $U \subset \mathbb{R}^n$ with $\mathcal{L}(\partial U) = 0$, show the following:

(a) The set $C^0(\overline{U})$ is a vector space.

(b) For $p \geq 1$, the following hold

- **(i)** $\|h\|_{L^p} \geq 0$ with equality if and only if h is the zero function in $C^0(\overline{U})$.
- **(ii)** $\|ch\|_{L^p} = |c| \; \|h\|_{L^p}$ for $c \in \mathbb{R}$ and $h \in C^0(\overline{U})$.
- **(iii)** $\|g + h\|_{L^p} = \|g\|_{L^p} + \|h\|_{L^p}$ for $g, h \in C^0(\overline{U})$.

(c) Use the L^p norm to attach a notion of **distance between functions** $d(g, h)$ for each pair of functions $g, h \in C^0(\overline{U})$ and each $p \geq 1$.

- **(i)** $d(g, h) \geq 0$ with equality if and only if $g = h$.
- **(ii)** $d(g, h) = d(h, g)$.
- **(iii)** $d(f, h) \leq d(f, g) + d(g, h)$.

(d) Show that a notion of **angle between functions** may be attached to a pair of nonzero functions $g, h \in C^0(\overline{U})$.

Exercise 1.21 Discuss cases of equality in parts **(b)(iii)** and **(c)(iii)** of Exercise 1.20.

Some care of a different flavor is required with respect to differentiation. First of all, differentiability at a point x in the open interval (a, b) as we know is defined by

$$h'(x) = \lim_{v \to 0} \frac{h(x+v) - h(x)}{v}. \tag{1.19}$$

Specifically, the function $h : (a, b) \to \mathbb{R}$ is said to be **differentiable at** $x \in (a, b)$ if the limit in (1.19) exists in the sense that there is some number $L \in \mathbb{R}$ for which given any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\left| \frac{h(x+v) - h(x)}{v} - L \right| < \epsilon \qquad \text{whenever} \qquad |v| < \delta. \tag{1.20}$$

If this condition holds, we write $h'(x) = L$. If $h : (a, b) \to \mathbb{R}$ is differentiable at every point $x \in (a, b)$, then we say h is **differentiable on the entire interval** (a, b). In this case, a function $h' : (a, b) \to \mathbb{R}$ is defined, and if this function is continuous, we write $h \in C^1(a, b)$. This is the basic condition defining what it means for a function to be **continuously differentiable** though the terminology is perhaps slightly opaque. Note carefully, however, that the definition just given for $C^1(a, b)$ applies for a, b **extended real numbers** with $a < b$. In particular, the values $a = -\infty$ and $b = +\infty$ are included/allowed.

Exercise 1.22 Define the vector space of functions $C^0(a, b)$ for $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$ with $a < b$, and show $C^1(a, b) \subset C^0(a, b)$.

An extension of the definition of differentiability at a point associated with (1.19) and (1.20) to the closed interval $[a, b]$ is readily obtained using the same approach used to define continuity above:

The function $h : [a, b] \to \mathbb{R}$ is said to be **differentiable at** $x \in [a, b]$ if the limit in (1.19) exists in the sense that there is some number $L \in \mathbb{R}$ for which given any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\left| \frac{h(x+v) - h(x)}{v} - L \right| < \epsilon \qquad \text{whenever} \qquad |v| < \delta \quad \textbf{and } x + v \in [a, b].$$

If this condition holds, we again write $h'(x) = L$, though often it may be convenient to denote the value of the derivative at an endpoint by

$$h'(a^+) = \lim_{v \searrow 0} \frac{h(a+v) - h(a)}{v} \qquad \text{or} \qquad h'(b^-) = \lim_{v \nearrow 0} \frac{h(b+v) - h(b)}{v}.$$

If $h : (a,b) \to \mathbb{R}$ is differentiable at every point $x \in [a,b]$, we say h is **differentiable on the entire interval** $[a,b]$. In this case, a function $h' : [a,b] \to \mathbb{R}$ is defined, and if $h' \in C^0[a,b]$ we write $h \in C^1[a,b]$ and say h is **continuously differentiable**.

Exercise 1.23 Show $C^1[a,b]$ is a vector space of functions, and the following statements are equivalent:

(i) $h \in C^1[a,b]$,

(ii) There exists some open interval $I \supset [a,b]$ and a function $g \in C^1(I)$ with

$$g\big|_{(a,b)} = h.$$

(iii) There exists a function $g \in C^1(\mathbb{R})$ with

$$g\big|_{(a,b)} = h.$$

The difference quotient formulation of the condition **(i)** in Exercise 1.23 does not carry over to higher dimensions. The alternative formulation **(ii)** allows the consideration of functions adequate for many purposes.[6]

Given an open set $U \subset \mathbb{R}^n$, a point $P \in U$, and $j \in \{1, 2, \ldots, n\}$, the **$j$-th partial derivative of $h : U \to \mathbb{R}$ at P** is defined by

$$\frac{\partial h}{\partial x_j}(P) = \lim_{v \to 0} \frac{h(P + v\mathbf{e}_j) - h(P)}{v}$$

when this limit exists. The value may also be denoted $D_j h(P)$, $h_{x_j}(P)$, or $D^{\mathbf{e}_j} h(P)$. If the j-th partial derivative of h exists at every point $P \in U$, then $D_j h : U \to \mathbb{R}$ is a well-defined function. If $D_j h : U \to \mathbb{R}$ is a well-defined function for each $j = 1, 2, \ldots, n$, then we say h is **partially differentiable** in all of U. Note: This is not the same as **differentiability** or the condition that h is **differentiable** even at a single point $P \in U$ which we define below. Partial differentiability is not the same (and does not imply in general) differentiability.

If h is partially differentiable and $D_j h \in C^0(U)$ for each $j = 1, 2, \ldots, n$, then we say h is **continuously partially differentiable** on all of U and write $h \in C^1(U)$.

[6]We are going to give a definition of $C^1(\overline{U})$ which is somewhat more restrictive than the one usually given.

Exercise 1.24 Show $C^1(U) \subset C^0(U)$.

If $U \subset \mathbb{R}^n$ is open and bounded and $h \in C^1(U)$, we say $h \in C^1(\overline{U})$ if the following condition(s) hold: There exists an open set $V \supset \overline{U}$ and a function $g \in C^1(V)$ for which

$$g\Big|_U = h.$$

Exercise 1.25 Let $h \in C^1(\overline{U})$.

(a) If there exists an open set $V \supset \overline{U}$ and functions $g, \tilde{g} \in C^1(V)$ for which

$$g\Big|_U = \tilde{g}\Big|_U = h,$$

then

$$g\Big|_{\overline{U}} = \tilde{g}\Big|_{\overline{U}}.$$

(b) Show

$$C^1(\overline{U}) = \left\{ g\Big|_U : g \in C^0(\overline{U}) \text{ and } g\Big|_U \in C^1(\overline{U}) \right\}.$$

(c) In view of Exercise 1.24 and parts **(a)** and **(b)** above, it makes sense to say $C^1(\overline{U}) \subset C^0(\overline{U})$ and in particular, $h \in C^0(\overline{U})$. (Explain.)

In view of the discussion above, we state what hopefully are considered natural geometry problems from calculus:

Newton's first differentiation problem (Problem 6): Given $h \in C^1[a, b]$, find the derivative $h' \in C^0[a, b]$.

Newton's general differentiation problem (Problem 6): Given an open and bounded set $U \subset \mathbb{R}^n$ and a function $h \in C^1(\overline{U})$, find the partial derivatives $D_j h \in C^0(\overline{U})$ for $j = 1, 2, \ldots, n$.

The following important definition was probably not known to Newton. It was certainly known to Fréchet (1878-1973).

Definition 1 Let U be an open set in $\mathbb{R}^n$, and let $h : U \to \mathbb{R}$ be a real valued function with domain U.

(i) The function $h : U \to \mathbb{R}$ is **differentiable at $\mathbf{x} \in U$** if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ for which

$$\lim_{|\mathbf{v}|\to 0} \frac{h(\mathbf{x}+\mathbf{v}) - h(\mathbf{x}) - L(\mathbf{v})}{|\mathbf{v}|} = 0.$$

The linear function $L : \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of h at $\mathbf{x}$ and is denoted by $dh_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$.

(ii) The function h is said to be **differentiable on all of** U if h is differentiable at each $\mathbf{x} \in U$.

(iii) We will say the function h is **differentiable on all of** $\overline{U}$ if there is some open set $V \subset \mathbb{R}^n$ with $V \supset \overline{U}$ and h is differentiable on all of V.

Exercise 1.26 Let U be a bounded open subset of $\mathbb{R}^n$, and let $h : U \to \mathbb{R}$ be a real valued function defined on U.

(a) Show that if h is differentiable at $\mathbf{x} \in U$, then h is continuous at $\mathbf{x} \in U$. Consequently, if h is differentiable on all of U, then $h \in C^0(U)$.

(b) Show that if h is differentiable at $\mathbf{x} \in U$, then h is partially differentiable at $\mathbf{x} \in U$.

(c) Show that if $h \in C^1(U)$, then h is differentiable on all of U.

(d) Show that if $h \in C^1(\overline{U})$ then h is differentiable on all of $\overline{U}$.

Newton's general differentiation problem (Problem 6a): Given an open set $U \subset \mathbb{R}^n$ and a function $h \in C^1(U)$, find the differential $dh_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ for each $\mathbf{x} \in U$.

Exercise 1.27 (challenge(s)) Let U be a open subset of $\mathbb{R}^n$ with $h \in C^1(U)$.

(a) Consider the function $g : U \times \mathbb{R}^n \to \mathbb{R}$ by

$$g(\mathbf{x}, \mathbf{v}) = dh_{\mathbf{x}}(\mathbf{v}).$$

Characterize the regularity of g?

(b) Consider the function $Dh : U \to \mathbb{R}^n$ by

$$Dh(\mathbf{x}) = \left(\frac{\partial}{\partial x_1}(\mathbf{x}), \frac{\partial}{\partial x_2}(\mathbf{x}), \dots, \frac{\partial}{\partial x_n}(\mathbf{x}) \right).$$

Characterize the regularity of Dh.

(c) Consider the function $\ell : U \to \daleth(\mathbb{R}^2)$ by

$$\ell(\mathbf{x}) = dh_{\mathbf{x}}$$

where $\daleth(\mathbb{R}^2)$ denotes the vector space of real valued linear functions $L : \mathbb{R}^2 \to \mathbb{R}$. Characterize the regularity of ℓ.

(d) If U is also bounded and $h \in C^1(\overline{U})$ under what conditions may it be said that the functions g, Dh, and ℓ considered above extend to $\overline{U}$, and what can be said about the regularity of each function in that case? Hint: Consider the case in which $U \subset \mathbb{R}^2$ and $\Gamma = \partial U$ is a cyclic path.

Once the delicate issue of determining how to say a function $h \in C^1(U)$ satisfies $h \in C^1(\overline{U})$ when $U \subset \mathbb{R}^n$ is an open set is addressed on one way or another, then the question of higher derivatives is more or less straightforward. For an open set $U \subset \mathbb{R}^n$ and for an integer $k \geq 2$, we say $h \in C^k(U)$ if all (partial) derivatives of order k or lower are in $C^0(U)$. All these partial derivatives are perhaps easiest to express in **multiindex notation**. A multiindex β is just an element of $\mathbb{N}_0^n$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ denotes the natural numbers with zero. Thus, a multiindex $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ has $\beta_j \in \mathbb{N}_0$ for $j = 1, 2, \dots, n$, and we write

$$D^\beta h = \frac{\partial h^{|\beta|}}{\partial^{\beta_1} x_1 \partial^{\beta_2} x_2 \cdots \partial^{\beta_n} x_n}$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. Thus, for $k = 1, 2, \dots$,

$$C^k(U) = \{h \in C^0(U) : D^\beta h \in C^0(U),\ |\beta| \leq k\}.$$

The vector space of functions $C^k(\overline{U})$ is defined inductively with

$$C^2(\overline{U}) = \{h \in C^1(\overline{U}) : D^\beta h \in C^1(\overline{U}),\ |\beta| = 1\}$$

with whatever definition is given to $C^1(\overline{U})$, and

$$C^k(\overline{U}) = \{h \in C^{k-1}(\overline{U}) : D^\beta h \in C^1(\overline{U}),\ |\beta| = k - 1\}$$

for $k = 2, 3, 4, \ldots$ in general.

We will mostly deal with derivatives of orders two and lower, so we will often use the more traditional notation. There is also a few other vector spaces that might be worth mentioning at this point. Given U an open set in $\mathbb{R}^n$,

$$C^\infty(U) = \bigcap_{k=0}^{\infty} C^k(U),$$

$$C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U}),$$

and $C^\omega(U)$ denotes the collection of **real analytic functions**, that is functions $h \in C^\infty(U)$ with the following property:

> For each $\mathbf{p} \in U$, there exists some $r > 0$ for which the series
>
> $$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta(\mathbf{p})}{\beta!} (\mathbf{x} - \mathbf{p})^\beta$$
>
> where $\beta! = \beta_1!\beta_2! \cdots \beta_n!$ and
>
> $$(\mathbf{x} - \mathbf{p})^\beta = (x_1 - p_1)^{\beta_1}(x_2 - p_2)^{\beta_2} \cdots (x_n - p_n)^{\beta_n}$$
>
> converges for $|\mathbf{x} - \mathbf{p}| < r$ and
>
> $$f(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta(\mathbf{p})}{\beta!} (\mathbf{x} - \mathbf{p})^\beta \qquad \text{for} \qquad |\mathbf{x} - \mathbf{p}| < r.$$

1.4.2 A new problem: curvature

Euclid's curvature problem (Problem 7): Given a circle of radius $r > 0$, what is the curvature of the circle?

This problem[7] is stated to suggest the formulation of the definition of curvature as much as the calculation of the particular number $k = 1/r$. What is curvature? In this case, a first approximation of the answer might be something like "Curvature is a number associated with a circle or a straight line which is zero for a straight line, positive for a circle, and decreasing with

[7] Perhaps first answered by Nicolas Oresme (c.1320–1382).

the radius for circles and tending to $+\infty$ as the radius decreases to zero." There are many choices of course, but the particular choice $k = 1/r$ with the limiting value

$$\lim_{r \nearrow \infty} \frac{1}{r} = 0$$

giving the curvature of a line (as a circle of infinite radius) is a natural one.

It is probably not natural to associate a curvature problem with Descartes and/or Fermat, but in honor of Fermat's triumph (and percociousness) in solving Descartes' calculus problem before he properly had the use of calculus, it may be natural to pose the following problem.

Exercise 1.28 Show the curvature of the folium of Descartes (Exercise 1.2) does not vanish.

Newton's first curvature problem (Problem 7): What is the curvature of the graph of function $h \in C^2[a, b]$ in the plane?

Exercise 1.29 Solve Newton's first curvature problem.

We should like to state a version of this problem for a function $\mathbf{x} : (a, b) \to \mathbb{R}^n$ which parameterizes a curve like, for example, the folium of Descartes. There is a problem.

Exercise 1.30 Plot the curve Γ determined by the function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ with $\mathbf{x} = (x_1, x_2)$ and

$$x_1(t) = t^3$$
$$x_2(t) = t^2.$$

We would say $\mathbf{x} \in C^\infty(\mathbb{R} \to \mathbb{R}^2)$ because $x_j \in C^\infty(\mathbb{R})$ for $j = 1, 2$. In fact, $x_j \in C^\omega(\mathbb{R})$ for $j = 1, 2$.

In view of the example of Exercise 1.30 it should be clear that differentiability (or regularity) alone is not enough to ensure that concept of curvature makes sense. Thus, we pause for a definition and an intervening technical question. Also, motivated by Exercise 1.30 it is natural to extend the notation for derivatives (at least for regular derivatives) to functions $\mathbf{x} : (a, b) \to \mathbb{R}^n$ defined on an open interval by

$$\mathbf{x}' = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right).$$

Definition 2 A function $\mathbf{x} \in C^2((a,b) \to \mathbb{R}^n)$ is said to be a **geometrically regular parameterization** of $\Gamma = \{\mathbf{x}(t) : t \in (a,b)\}$ if $|\mathbf{x}'(t)| \neq 0$ for $t \in (a,b)$.

Newton's first reparameterization problem (Problem 8): Given a geometrically regular parameterization $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$ of

$$\Gamma = \{\mathbf{x}(t) : t \in (a,b)\},$$

find a function $\gamma \in C^1((c,d) \to \mathbb{R}^n)$ defined on some interval $(c,d) \in \mathbb{R}$ and satisfying

(i) $|\gamma'(t)| \equiv 1$ for $t \in (c,d)$, and

(ii) $\Gamma = \{\gamma(t) : t \in (c,d)\}$.

Exercise 1.31 Solve Newton's first reparameterization problem.

A parameterization $\gamma \in C^1((a,b) \to \mathbb{R}^n)$ satisfying the conditions in Newton's first reparameterization problem is called a **parameterization by arclength**.

Newton's second curvature problem (Problem 7): What is the curvature of of a curve given by a geometrically regular parameterization $\mathbf{x} \in C^2((a,b) \to \mathbb{R}^n)$?

Newton's third curvature problem (Problem 7): Does the curvature of of a curve given by a geometrically regular parameterization $\mathbf{x} \in C^2((a,b) \to \mathbb{R}^2)$ determine the curve?

Exercise 1.32 Find two very different curves Γ_1 and Γ_2 determined by arclength parameterizations

$$\gamma_1 \in C^2((-1,1) \to \mathbb{R}^3) \qquad \text{and} \qquad \gamma_2 \in C^2((-1,1) \to \mathbb{R}^3)$$

and satisfying the following:

(i) $\gamma_1(0) = \gamma_2(0)$,

(ii) $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$ where

$$\dot{\gamma}_j = \frac{d\gamma_j}{ds},$$

and

(iii) $\ddot{\gamma}_1(s) \equiv \ddot{\gamma}_2(s)$ for $-1 < s < 1$ where

$$\ddot{\gamma}_j = \frac{d^2\gamma_j}{ds^2}.$$

Newton's fourth curvature problem (Problem 7): Are there quantities similar to curvature determining a curve given by a geometrically regular parameterization $\mathbf{x} \in C^2((a,b) \to \mathbb{R}^n)$?

This last problem should probably not be attributed to Newton but rather to someone like Jean Frédéric Frenet (1816–1900) or Joseph-Alfed Serret (1819–1885) both of whom were asking this kind of question sometime around 1850.

It makes sense to extend at least some of these kinds of problems to higher dimensional (parameterized) objects at least for graphs.

Newton's fifth curvature problem (Problem 7): What is the curvature of the graph of a function $h \in C^2(U \to \mathbb{R})$ where U is an open subset of $\mathbb{R}^2$?

Of course, this should really probably be called **Gauss's first curvature problem**, but certainly it is a question Newton might have thought about.

Newton's sixth curvature problem (Problem 7): What is the curvature of the graph of a function $h \in C^2(U \to \mathbb{R})$ where U is an open subset of $\mathbb{R}^n$?

And of course this would qualify as **Riemann's first curvature problem**.

1.4.3 Weierstrass: minimization of functionals

Contemplating Newton's (and Leibniz') calculational revolution, Karl Weierstrass undertook the task of imposing some additional rigour in the application of the suggested techniques. Much of this heightened level of critical thought is familiar to us now, but certainly at the time Weierstrass was asking questions that might have never occurred to Newton or Leibniz or even Weierstrass' contemporaries. I do not know if the following problem was originally posed by Weierstrass, but he certainly knew about it.

Weierstrass' minimization problem (Problem 5): Among all C^1 graphs, with endpoints fixed at (a, y_a) and (b, y_b) as in Newton's length problem, which graph has the shortest length?

It may seem audacious or even absurd to pose such a problem. Of course,

$$h_0(x) = \frac{y_b - y_a}{b-a}\ (x-a) + y_a = \frac{y_b(x-a) + y_a(b-x)}{b-a} \tag{1.21}$$

gives the solution, but asking such a question turns out to be an important one as we will see below. For the moment, let's humor eccentric Professor Weierstrass and try to answer his question as rigorously as we can (i.e. with as much critical thinking as we can muster).

We have a set of admissible functions

$$\mathcal{A} = \{h \in C^1[a,b] : h(a) = y_a \text{ and } h(b) = y_b\}$$

and a real valued functional length : $\mathcal{A} \to [0, \infty)$ by

$$\text{length}[h] = \int_a^b \sqrt{1 + [h'(x)]^2}\, dx. \tag{1.22}$$

I've used the same name for the length functional in the special case of graphs under consideration at the moment rather than the more general length functional

$$\text{length}[\alpha] = \int_{(a,b)} |\alpha'| \tag{1.23}$$

given below (3.2) in Chapter 3 and which may be familiar from elementary differential geometry.

Exercise 1.33 Show the length functional (1.22) arises as a special case of the length functional (1.23) which applies to the length of paths parameterized by a function $\alpha \in C^1([a,b] \to \mathbb{R}^2)$.

Exercise 1.34 Specify an appropriate admissible class corresponding to parameterized paths on which to consider a more general version of Weierstrass' minimization problem (Problem 5) involving the length functional given in (1.23).

Notice that if $h \in \mathcal{A}$ and $\phi \in C^1[a,b]$ with $\phi(a) = \phi(b) = 0$, then for every $t \in \mathbb{R}$, there holds $h + t\phi \in \mathcal{A}$. Thus, if

$$\text{length}[h] \le \text{length}[g] \qquad \text{for every } g \in \mathcal{A},$$

then in particular, $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \text{length}[h + t\phi]$ has a minimum value at $t = 0$. Assuming the function $f \in C^1(\mathbb{R})$, it follows that a minimizer h must satisfy

$$f'(0) = 0.$$

Exercise 1.35 Show that if

$$h \in \mathcal{A} = \{g \in C^1[a,b] : g(a) = y_a \text{ and } g(b) = y_b\}$$

and

$$\phi \in C_0^1[a,b] = \{g \in C^1[a,b] : g(a) = 0 = g(b)\},$$

then $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \text{length}[h + t\phi]$$

satisfies $f \in C^1(\mathbb{R})$ and calculate $f'(t)$ for each $t \in \mathbb{R}$.

The calculation suggesed in Exercise 1.35 should lead to an expression of the form

$$f'(t) = \int_a^b [h'(x) + t\phi'(x)]\ \Phi(h'(x) + t\phi'(x))\ \phi'(x)\, dx$$

for some function $\Phi \in C^\infty(\mathbb{R})$. If h is taken as a minimizer of the length functional length : $C^1[a,b] \to [0,\infty)$ as suggested above, then we should have $f'(0) = 0$. That is,

$$\int_a^b h'(x)\ \Phi(h'(x))\ \phi'(x)\, dx = 0 \tag{1.24}$$

with the equality holding for every

$$\phi \in C_0^1[a,b] = \{f \in C^1[a,b] : f(a) = 0 = f(b)\}.$$

Thus, we contemplate the question:

> What does an integral equality like that in (1.24) imply about the factor $h'\Phi(h')$ in the integrand, if it holds for all $\phi \in C_0^1[a,b]$?

The first thing I would like to point out is the following: This question would be a lot easier if there was no derivative on the function ϕ in (1.24).

The expression on the left in (1.24) defines a linear function $L : C_0^1[a,b] \to \mathbb{R}$ by

$$L[\phi] = \int_{(a,b)} h'\Phi(h')\ \phi', \tag{1.25}$$

and the question is: What does the condition $L[\phi] = 0$ for all $\phi \in C_0^1[a,b]$ say about the factor $h'\Phi(h')$ in the integrand?

The easier version referered to above is this: Let $M : C_0^0[a,b] \to \mathbb{R}$ be given by

$$M[\phi] = \int_a^b g(x)\ \phi(x)\, dx \tag{1.26}$$

where $g \in C^0[a,b]$ is some (fixed) continuous function and

$$C_0^0[a,b] = \{f \in C^0[a,b] : f(a) = 0 = f(b)\}.$$

If $M[\phi] = 0$ for all $\phi \in C_0^0[a,b]$, then $g(x) = 0$ for $x \in [a,b]$. This assertion[8] is called the **fundamental lemma of the calculus of variations** or the **fundamental lemma of vanishing integrals**, and it is easy to prove.

The functions ϕ appearing in definitions of integral operators like those defined in (1.25) and (1.26) have become known as **test functions**.

Proof of the fundamental lemma of vanishing integrals: Assume $g(x_0) > 0$ for some $x_0 \in (a,b)$. By the continuity of g, there is some $\delta > 0$ with $\delta < \min\{x_0 - a, b - x_0\}$ and

$$g(x) \geq \frac{g(x_0)}{2} > 0 \qquad \text{for } |x - x_0| < \delta. \tag{1.27}$$

Exercise 1.36 Actually use continuity (pick an ϵ and then get an $\delta > 0$ so that $|g(x) - g(x_0)| < \epsilon$ for $|x - x_0| < \delta$) and use the triangle inequality to conclude/prove (1.27) holds.

[8]Technically, this is only one version of the fundamental lemma of the calculus of variations. All results which might be called this have roughly the same form. Some versions are "stronger" because they either allow more general functions g to play the role of the integrand factor in (1.26) and/or they require the vanishing of the integral functional M to hold for smaller classes of test functions ϕ.

Once we have a fixed $\delta > 0$ as described above, we can note that $\phi : [a.b] \to [0, \delta_0]$ by

$$\phi_0(x) = \chi_{[x_0-\delta, x_0+\delta]}(x)\,[\delta - |x - x_0|] \tag{1.28}$$

satisfies $\phi_0 \in C_0^0[a, b]$. Here I have used a **characteristic function** χ. Such functions are also sometimes called **indicator functions**. In case you are not familiar, given any set X and a subset $A \subset X$, the characteristic function with support on A is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Characteristic functions are typically discontinuous, but in this case things work out.

Exercise 1.37 Show the test function ϕ_0 defined in (1.28) satisfies the following:

(a) $\phi_0 \in C_0^0[a, b]$.

(b) $\phi_0(x) \geq 0$ for $a \leq x \leq b$.

(c) $\phi_0(x) > 0$ for $|x - x_0| < \delta$.

(d) $\phi_0(x) \equiv 0$ for $|x - x_0| \geq 0$.

It follows that

$$\begin{aligned} M[\phi_0] &= \int_{(a,b)} g\ \phi_0 \\ &= \int_{(x_0-\delta, x_0+\delta)} g\ \phi_0 \\ &\geq \frac{g(x_0)}{2} \int_{(x_0-\delta, x_0+\delta)} \phi_0 \\ &= g(x_0)\ \delta^2 \\ &> 0. \end{aligned}$$

This contradicts the assumption $g(x_0) > 0$ at some $x_0 \in (a, b)$. We conclude $g(x) \leq 0$ for $x \in (a, b)$.

Exercise 1.38 Give a detailed argument showing $g(x) \geq 0$ for $x \in (a, b)$.

From Exercise 1.38 one can conclude $g(x) \equiv 0$ for $a < x < b$. By continuity

$$g(a) = \lim_{x \searrow a} g(x) = 0 \qquad \text{and} \qquad g(b) = \lim_{x \nearrow b} g(x) = 0.$$

Therefore $g(x) \equiv 0$ for $x \in [a, b]$ as claimed, and the proof is complete. $\square$

So we have some version of the fundamental lemma of vanishing integrals which applies to functionals of the form given in (1.26). Unfortunately, the functional $L : C_0^1[a, b] \to \mathbb{R}$ appearing in (1.25) does not quite have this form, and this is the functional with which we have to deal. Here is a first observation about this more restrictive condition: If the factor $h'\Phi(h')$ is continuously differentiable, that is **if**

$$h'\Phi(h') \in C^1[a, b],$$

then we can integrate by parts to write the value of $L[\phi]$ as

$$\begin{aligned} L[\phi] &= h'\Phi(h')\ \phi\Big|_{x=a}^{b} - \int_{(a,b)} \frac{d}{dx}[h'\Phi(h')]\ \phi \\ &= -\int_{(a,b)} \frac{d}{dx}[h'\Phi(h')]\ \phi. \end{aligned}$$

Then taking the integrand factor $g = -(h'\Phi(h'))'$ in the fundamental lemma we can conclude $(h'\Phi(h'))' = 0$ or $h'\Phi(h') = c$ is constant.

Exercise 1.39 Go ahead and solve the two point boundary value problem

$$\begin{cases} h'\Phi(h') = c, & a \le x \le b \\ h(a) = y_a, \\ h(b) = y_b \end{cases}$$

for the ordinary differential equation $h'\Phi(h') = c$ where c is an (unknown) constant to be determined along with the function $h \in C^1[a, b]$. Hint: You should get the function given in (1.21).

So this gives what we expect (and presumably want), but there are a few details to clean up. One big problem is that we essentially assumed a minimizer h must satisfy $h \in C^2[a, b]$. The assumption lead to precisely the minimizers we expected, but it is still troubling that we had to make the assumption without any real justification. There are various ways to deal with this problem which generally fall under the heading of the **lemma of Dubois-Reymond**. Here is one version:

Lemma 1.1 Let $g \in C^0[a,b]$ and $L : C_0^1[a,b] \to \mathbb{R}$ be given by

$$L[\phi] = \int_{(a,b)} g\ \phi'.$$

If $L[\phi] = 0$ for all $\phi \in C_0^1[a,b]$, then $g \equiv c$ is constant (for some constant c).

Proof: We will use the fundamental lemma which applies to linear functionals having the form satisfied by $M : C_0^0[a,b] \to \mathbb{R}$ discussed above. The question to ask (though maybe it is not an obvious one) is the following:

> How can you construct a C_0^1 function from an arbitrary C_0^0 function?

Let $\psi \in C_0^0[a,b]$. The function ψ may not be differentiable, but ψ is good for integration. Notice also, that if it happens that $\psi = \phi'$ for some $\phi \in C_0^1[a,b]$, then by the fundamental theorem of calculus

$$\int_{(a,b)} \psi = \int_{(a,b)} \phi' = \phi(b) - \phi(a) = 0.$$

Thus, a necessary condition for $\psi \in C_0^0[a,b]$ to satisfy $\psi = \phi'$ for some $\phi \in C_0^1[a,b]$ is

$$\int_{(a,b)} \psi = 0. \tag{1.29}$$

It turns out this is also sufficient.

Exercise 1.40 Show that if $\psi \in C_0^0[a,b]$ and (1.29) holds, then there is some $\phi \in C_0^1[a,b]$ with $\psi = \phi'$.

Let $\mu \in C_0^0[a,b]$ satisfy

$$\int_{(a,b)} \mu = 1.$$

Such a function should be easy to find. For example, the function $\mu = \phi_0/(2\delta^2)$ where ϕ_0 is defined in (1.28) should work.

Given an arbitrary $\psi \in C_0^0[a,b]$, consider the function

$$\psi - C\mu$$

where C is a constant. The constant C is **not** going to be the constant in the statement of the lemma. In fact, take

$$C = \int_{(a,b)} \psi.$$

Then we see

$$\int_{(a,b)} (\psi - C\mu) = \int_{(a,b)} \psi - C \int_{(a,b)} \mu = \int_{(a,b)} \psi - C.$$

In view of the value of C this quantity vanishes, which means the function $\psi - C\mu$ is the derivative of some function $\phi \in C_0^1[a,b]$. In particular, this is a function for which the conditions of the lemma hold so that

$$L[\psi - C\mu] = 0.$$

Since L is linear, and C is a constant, this also means that for every $\psi \in C_0^0[a,b]$ we have $L[\psi] - CL[\mu] = 0$. That is,

$$\int_{(a,b)} g\ \psi - C \int_{(a,b)} g\ \mu = 0. \tag{1.30}$$

Now remember that g is a fixed function, and μ is a fixed function too, so whatever form the value

$$c = L[\mu] = \int_{(a,b)} g\ \mu$$

happens to take, it is a constant, and this **is** the constant in the statement of the lemma. Also, we remember again the value of the (other) constant C so that (1.30) becomes

$$\int_{(a,b)} g\ \psi - c \int_{(a,b)} \psi = 0 \qquad \text{for all } \psi \in C_0^0[a,b].$$

That is,

$$\int_{(a,b)} (g - c)\ \psi = 0 \qquad \text{for all } \psi \in C_0^0[a,b].$$

By the fundamental lemma then we have the lemma of Dubois-Reymond: $g \equiv c$. $\square$

The preceeding result has a rather interesting corollary. A function $h \in \mathcal{A}$ for which the condition

$$L[\phi] = \int_{(a,b)} h'\Phi(h')\ \phi' = 0 \qquad \text{for all } \phi \in C_0^1[a,b]$$

holds with a linear operator L like the one in (1.25) is called a C^1 **weak extremal** for the problem. Every minimizer of the length functional, for example, is a C^1 weak extremal, but coneivably there could be other such functions which are not minimizers. It turns out in this case, that all C^1 weak extremals h actually satisfy $h \in C^\infty[a,b]$. There are other problems, however, for which there exist C^1 weak extremals which are really only in C^1, say in $C^1[a,b]$ and not in $C^2[a,b]$. The Lemma of Dubois-Reymond applies in all these cases to imply that whatever factor turns up in the integrand will, if necessary, somehow make the resulting expression differentiable.

Corollary 1.1 If $h \in \mathcal{A}$ is a C^1 weak extremal for length, then the expression $h'\Phi(h')$ which is (on the face of it) only continuous ($h'\Phi(h') \in C^0[a,b]$) is actually continuously differentiable ($h'\Phi(h') \in C^1[a,b]$) and

$$(h'\Phi(h'))' = 0.$$

Thus in fact $\phi\Phi(h') \in C^\infty[a,b]$.

Exercise 1.41 Ignore the spatial restriction of $B_1(\mathbf{0})$ in relation to the Riemannian manifold $\mathcal{B}$ and find some C^1 weak extremals for $\text{length}_{\mathcal{B}} : \mathcal{A} \to [0,\infty)$ where $-1 < a < b < 1$, $a^2 + y_a^2, b^2 + y_b^2 < 1$,

$$\mathcal{A} = \{h \in C^1[a,b] : h(a) = y_a, h(b) = y_b\},$$

and

$$\text{length}_{\mathcal{B}}[h] = \int_{x\in(a,b)} \frac{4}{4 + x^2 + h^2}\ \sqrt{1 + h'^2}$$

is obtained by using the parameterization $\alpha(t) = (t, h(t))$.

This question was considered by Ty Bondurant, and some of what he is found is included in Appendix B below. For the moment, I'm going to move on to the other big problem with the discussion above,[9] and then to a generalization of Weierstrass' (in some ways) irritating question.

[9] I should emphasize that if you have not completed or at least seriously attempted Exercise 1.41, then the best immediate use of your time to make progress toward understanding Riemannian manifolds may be to think about that exercise.

When given a problem that involves minimizing a functional like

$$\text{length} : \mathcal{A} \to [0, \infty)$$

with domain a subset $\mathcal{A}$ of an infinite dimensional space of functions like $C^1[a,b]$, it is not always clear that a minimizer **exists**. Everything we have said above leading to the ordinary differential equation (called the **Euler-Lagrange equation** for C^1 weak extremals (and minimzers in particular) is based on the assumption that a minimizer exists. In calculus, for example if we look at a particular function $h \in C^0[a,b]$, then the **extreme value theorem** guarantees there exist points x_0 and x_1 in $[a,b]$ with

$$h(x_0) = \min_{x \in [a,b]} h(x) \qquad \text{and} \qquad h(x_1) = \max_{x \in [a,b]} h(x).$$

In short, we do not have such a theorem for functionals on subsets of infinite dimensional spaces in general. One says that in general there is no **compactness**. Fortunately, this question of existence can be addressed directly in the case of Euclidean distance in Weierstrass' minimization problem (Problem 5) as suggested below.

Consider the minimization of

$$\text{length}[\alpha] = \int_{(a,b)} |\alpha'|$$

over

$$\mathcal{A} = \{\alpha \in C^1([a,b] \to B_1(\mathbf{0})) : \alpha(a) = \mathbf{x},\ \alpha(b) = \mathbf{y}\}.$$

This is essentially

Weierstrass' second minimization problem (Problem 5): Among all geometrically regular paths parameterized by a function $\alpha \in C^1([a,b] \to \mathbb{R}^2)$ and satisfying $\alpha(a) = \mathbf{x}$ and $\alpha(b) = \mathbf{y}$ for some fixed $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, find the path of minimum length.

Exercise 1.42 Show

$$\text{length}[\alpha] = \int_{(a,b)} |\alpha'| \geq |\mathbf{y} - \mathbf{x}| \qquad \text{for } \alpha \in \mathcal{A}.$$

Hint: Look ahead at Exercises 1.7-1.9 and use the triangle inequality.

Exercise 1.43 Consider the assertion of Exercise 1.42:

(a) Apply the assertion to conclude the nonparametric problem has a solution (minimizer) $h \in C^\infty[a,b]$.

(b) Apply the assertion to conclude the parametric problem has a solution $\alpha \in C^\infty([a,b] \to \mathbb{R}^2)$.

(c) Apply the Lemma of Dubois-Reymond to conclude **every** minimizer for the nonparametric problem is given by (1.21).

Let us again assume we have a minimizer $\alpha \in \mathcal{A}$ of the functional

$$\text{length} : \mathcal{A} \to [0, \infty).$$

Now, we take a function $\phi \in C^1([a,b] \to \mathbb{R}^2)$ with $\phi(a) = \mathbf{0} = \phi(b)$ so that $\alpha + t\phi \in \mathcal{A}$ for each $t \in \mathbb{R}$, and we consider the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \text{length}[\alpha + t\phi] = \int_{(a,b)} |\alpha' + t\phi'|.$$

Since α is assumed to minimize length, we should have $f'(0) = 0$. Thus, we calculate

$$f'(t) = \int_{(a,b)} \frac{\langle \alpha' + t\phi', \phi' \rangle_{\mathbb{R}^2}}{|\alpha' + t\phi'|}$$

and

$$f'(0) = \int_{(a,b)} \frac{\langle \alpha', \phi' \rangle_{\mathbb{R}^2}}{|\alpha'|} = 0.$$

Of course, there is a (potential) problem here when $|\alpha' + t\phi'| = 0$, but the assumption that α is a geometrically regular parameterization should take care of this at some level. Note the product formula:

$$\frac{d}{dt} \left\langle \frac{\alpha'}{|\alpha'|}, \phi \right\rangle_{\mathbb{R}^2} = \left\langle \left(\frac{\alpha'}{|\alpha'|} \right)', \phi \right\rangle_{\mathbb{R}^2} + \frac{\langle \alpha', \phi' \rangle_{\mathbb{R}^2}}{|\alpha'|}.$$

This allows us to integrate by parts (if we assume some extra regularity for the minimizer α in addition to possibly the nonvanishing of $|\alpha'|$. Proceeding as if all this is no problem:

$$f'(0) = \frac{\langle \alpha', \phi \rangle_{\mathbb{R}^2}}{|\alpha'|} \Big|_{t=a}^{b} - \int_{(a,b)} \left\langle \left(\frac{\alpha'}{|\alpha'|} \right)', \phi \right\rangle_{\mathbb{R}^2}.$$

If this quantity vanishes we can consider the component functions $\phi_j \in C^1[a,b]$ of $\phi = (\phi_1, \phi_2)$ for $j = 1, 2$ separately (taking for example $\phi_2 \equiv 0$ and ϕ_1 arbitrary) to conclude the coupled system of ordinary differential equations

$$\begin{cases} (\alpha_1/|\alpha'|)' &= 0 \\ (\alpha_2/|\alpha'|)' &= 0 \end{cases}$$

should hold for a minimizer. The vector quantity in this case (assuming α is geometrically regular according to Definition 2 above)

$$\frac{1}{|\alpha'|} \left(\frac{\alpha'}{|\alpha'|} \right)' \tag{1.31}$$

is called the **curvature vector**, so the necessary condition we have obtained for a minimizer is that the curvature vanishes.

Exercise 1.44 Explain why the expression in (1.31) is a reasonable expression to call a/the curvature vector, and determine all paths with vanishing curvature vector.

Perhaps now you can address the following:

Weierstrass' general length minimization problem (Problem 5): Among all geometrically regular paths parameterized by a function $\alpha \in C^1([a,b] \to \mathbb{R}^n)$ and satisfying $\alpha(a) = \mathbf{x}$ and $\alpha(b) = \mathbf{y}$ for some fixed $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, find the path of minimum length.

We conclude with a problem that should make sense if you have had a look at Chapter 3.

Riemann's length minimization problem (Problem 5): Among all geometrically regular paths parameterized by a function $\alpha \in C^1([a,b] \to \mathbb{R}^n)$ and satisfying $\alpha(a) = \mathbf{x}$ and $\alpha(b) = \mathbf{y}$ for some fixed $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, find the path that minimizes

$$\text{length}_M[\alpha] = \int_{(a,b)} \sqrt{\langle (g_{ij})\ \alpha', \alpha' \rangle_{\mathbb{R}^n}}$$

where (g_{ij}) is an $n \times n$ positive definite symmetric matrix having entries g_{ij} for $i, j = 1, 2, \ldots, n$ satisfying $g_{ij} \in C^0(\mathbb{R}^2)$.

Exercise 1.45 Derive a system of ordinary differential equations for the component functions α_j, $j = 1, 2, \ldots, n$ for a minimizer of Riemann's length minimization problem. (Assume as much regularity as you need, but note how much regularity that is. Specifically, explain how regular the component functions α_j, $j = 1, 2, \ldots, n$ and the matrix entries g_{ij}, $i, j = 1, 2, \ldots, n$ need to be in order for your calculus techniques/manipulations to be justified.)