Appendix D

Questions

This appendix has some of the character of the previous appendix in that it records some student questions and attempts to explain various aspects of them and especially (re)formulate them more precisely when that seems appropriate. The difference is that the previous appendix was nominally limited to thoughts and questions related to the special example manifold $\mathcal{B}$. Some of the topics I attempt to address below were also the result of questions related to the discussion of $\mathcal{B}$ in class, but they seemed to me to range in some way farther afield from the main aspects of that discussion. Some are more or less general questions which may be viewed as entirely unrelated to the details associated with $\mathcal{B}$.

The discussion in class usually extended beyond what I've recorded (and formulated) below, and I am composing the material below at a distance from those class meetings. Partially due to that distance in time and partially due to the fact that I may not have understood the question entirely or misinterpreted it in some way, there may be some innaccuracies of one sort or another. For this I apologize. In any case, I had written something in my notes or had something in my mind which I thought might represent an interesting direction of inquiry or about which I had something to say/write and wanted to record something about it here for future reference.

D.1 Other student questions

D.1.1 John Stavroulakis' question about geodesics c. March 19, 2024

To begin with a simple instance, the length functional $L : C^1([a,b] \to \mathbb{R}^n) \to [0,\infty)$ (for parametric paths in $\mathbb{R}^n$)

$$L[\mathbf{x}] = \int_a^b |\mathbf{x}'(t)|\, dt$$

considered on the admissible class

$$\mathcal{A} = \left\{ \mathbf{x} \in C^1([a,b] \to \mathbb{R}^n) : \mathbf{x}(a) = \mathbf{p},\ \mathbf{x}(b) = \mathbf{q},\ \frac{1}{|\mathbf{x}'|} \in L^1(a,b) \right\}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ with $\mathbf{p} \neq \mathbf{q}$ has first variation $\delta L_{\mathbf{x}} : C_c^\infty((a,b) \to \mathbb{R}^n) \to \mathbb{R}$ with values given by

$$\delta L_f[\vec{\phi}] = \int_a^b \frac{\langle \mathbf{x}'(t), \vec{\phi}'(t) \rangle_{\mathbb{R}^n}}{|\mathbf{x}'(t)|}\, dt$$

displays an obvious singularity associated with functions for which $\mathbf{x}'(t)$ takes the value $\mathbf{0} \in \mathbb{R}^n$. The length functional for paths $\mathbf{x} \in C^1([a,b] \to M)$ taking values in a manifold M displays the same kind of singular behavior in its first variation, and one can ask generally, what role is played by this singularity in the attempt to identify and analyze minimizers/geodesics?

In our discussion of geodesics Ruijia Cao attempted to consider relations between the length functional and the Dirichlet energy, which in the simple case above takes the form

$$\mathcal{D}[\mathbf{x}] = \int_a^b |\mathbf{x}'(t)|^2\, dt.$$

It may be noted that $\mathcal{D}$ has a nice nonsingular first variation on the larger admissible class

$$\mathcal{A}_{\mathcal{D}} = \left\{ \mathbf{x} \in C^1([a,b] \to \mathbb{R}^n) : \mathbf{x}(a) = \mathbf{p},\ \mathbf{x}(b) = \mathbf{q} \right\}$$

with no restriction concerning integrability of $1/|\mathbf{x}'|$. With this in mind, I note the following:

1. There seems to be very little discussion in the literature concerning "singular Lagrangians." More specifically, there are many books and papers about the attempted minimization of integral functionals $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ for functionals
$$\mathcal{F}[\mathbf{x}] = \int_a^b F(t, \mathbf{x}(t), \mathbf{x}'(t))\, dt. \tag{D.1}$$
These generally fall under the heading of the calculus of variations. The function $F : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ appearing in the integrand of the functional in (D.1) is called the **Lagrangian** of the functional, and the Lagrangian of an integral functional is very often assumed to be smooth and satisfy, for example, $F \in C^\infty([a, b] \times \mathbb{R}^n \times \mathbb{R}^n)$. This condition very noticeably fails for length functionals but does hold for Dirichlet energy.

   Given the evidently very important example of length functionals associated with geodesics, it seems somewhat striking to me that there seems to be little formal discussion of "singular Lagrangians" in the calculus of variations.

2. The usual approach does seem to be to consider the Dirichlet energy instead and prove results along the following lines:

   (a) The Dirichlet energy has well-defined minimizers in a (large) admissible class $\mathcal{A}_\mathcal{D}$.

   (b) Minimizers of Dirichlet energy in $\mathcal{A}_\mathcal{D}$ have a "built-in" requirement that they fall into a more restricted admissible class $\mathcal{A}$ of functions satisfying an integrability condition. Specifically, extremals of Dirichlet energy, and minimizers in particular, must satisfy some condition along the lines of $|\mathbf{x}'|$ is a nonzero constant.

   (c) Extremals of Dirichlet energy in $\mathcal{A}_\mathcal{D}$ are also extremals of length within $\mathcal{A}$.

   (d) The convexity properties of Dirichlet energy are easier to deal with than those of the length functional, so usually minimizers of Dirichlet energy are considered instead of minimizers of length directly.

3. Generally to pursue these developments, as they are addressed in many textbooks on differential geometry, one must understand the intrinsic

differentiation of vector fields at a level beyond what we were able to cover/achieve this semester. I'm leaving this as a placeholder to come back and offer my own exposition of these relations between Dirichlet energy and the length functional at a later time. The standard approach seems to me to be relatively adequate in many respects, but I still think it would be (or could be) very interesting to have some kind of general theory of "singular Lagrangians" which seems at the current time (as far as I can tell) to be lacking in the literature.

D.2 Lu Li's question on geodesics Wednesday April 17, 2024

This was asked rather later than the questions below but relates to a certain extent to John Stavroulakis' question above, so I've placed it here. The question (which was also asked in some form by others—perhaps Ruijia Cao) is: How do you prove a path which is a solution of the geodesic equation is a unique length minimizer? This question arose in the context of my mentioning global results/theorems asserting the existence of conjugate points, that is pairs of points P and Q on a manifold M having the property that there does **not** exist a unique length minimizing path connecting P and Q.

The roughest answer is through convexity techniques in the calculus of variations, that is to say there are results asserting that certain integral functionals $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ are convex and critical points, that is for example weak extremals like geodesics, are indeed minimizers. It should be mentioned, in addition that the functional to which these techniques are usually applied is a/the Dirichlet energy (functional). With the comments above, the basic approach is along the following lines:

1. Every path which is a solution of the geodesic equations (whose existence is obtained by using the Dirichlet energy and comes from that discussion equipped with a constant speed parameterization) can be considered with a constant speed parameterization and is then also critical for Dirichlet energy.

2. Convexity along with existence and uniqueness of minimizers of the Dirichlet energy on some restricted admissible class $\mathcal{A}_0$, say restricting to paths in some open submanifold M_0 of points Q where one has also

$P \in M_0$, is associated with the existence of certain non-vanishing vector fields called Jacobi fields. This material can be found in chapters 3 and 5 of [2].

3. A point of vanishing, or a singular point, in a Jacobi field gives the existence of conjugate points, and one approach to obtaining such vanishing is through the theorem saying there exist no non-vanishing vector fields on compact manifolds homeomorphic to a sphere. This theorem is said to assert that "one cannot comb the hair on a billiard ball."

D.3 Lu Li's question on second derivatives

After discussing the intrinsic derivative(s) of a real valued function $f : M \to \mathbb{R}$ defined on a Riemannian manifold which were determined naturally to be directional derivatives, Lu Li asked about the possibility of intrinsic second derivatives. These are natural objects to consider in some way, shape, or form. However, certain things should be expected:

1. As there are no specified basis vectors, analogous to the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, in/on many n-dimensional (Riemannian) manifolds M, a direct analogue of second partial derivatives

$$D_{ij} f = D^{\mathbf{e}_i + \mathbf{e}_j} f = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

should **not** be expected.

2. Even the full first intrinsic derivative however one wishes to describe it, for example as a gradient field Df or as a differential map/tensor df is a little bit obscure(d) as either determined by inner products

$$\mu_P(Df(P), z) = df_P(z) = \lim_{t \to t_0} \frac{f \circ \alpha(t) - f(P)}{t} = \lim_{t \to t_0} \frac{f \circ \mathbf{p}(\mathbf{x} + td\xi_P(z)) - f(P)}{t}$$

where $z = [\alpha]$ and/or depending on a filament path or a path in a chart. Thus, one should expect an intrinsic Hessian $D^2 f$ or full second derivative to be some kind of relatively complicated object.

3. In particular, since the full first derivative Df is a vector field, one should expect to have a good understanding of the full derivative $\nabla(Df)$ of a vector field in order to understand the intrinsic Hessian $D^2 f = \nabla(Df)$.

D.4 Sergey Blinov's question on linear-Leibnizian operators

There is a space of linear-Leibnizian operators $\mathcal{LL}_P M$ that is linear space isomorphic to $\mathcal{L}_P M$, and there is also a $cC^\infty(M)$ module $\mathscr{X}\mathscr{X}(M)$ consisting of linear-Leibnizian fields that is module-linear isomorphic to $\mathscr{X}(M)$ the module of vector fields on M. If we use the capital letter W to denote the linear-Leibnizian field corresponding to the vector field $w \in \mathscr{X}(M)$, so that $W : cC^\infty(M) \to cC^\infty(M)$ with

$$W[gh] = W[g]\ h + g\ W[h]$$

for $g, h \in cC^\infty(M)$ and $W_P : cC^\infty(M) \to \mathbb{R}$ by

$$W_p[g] = W[g](P)$$

has $W_P \in \mathcal{L}(cC^\infty(M))$, then is there a way to realize $\mathcal{LL}_P M$ and/or $\mathscr{X}\mathscr{X}(M)$ as a linear dual space?

I don't really know the answer to this question, but I did find an assertion to the effect that "The set of all derivations at a point P on a manifold can be thought of as the dual space of J/J^2, where J is the maximal ideal of smooth functions vanishing at P." If this sort of thing makes sense to you, then perhaps it gives or leads to some kind of answer. I guess something like this should be true, but it's probably complicated (and perhaps not very useful as suggested by other people who posted in the same thread).

> https://mathoverflow.net/questions/117374/why-is-the-leibniz-rule-a-definition-for-derivations

D.5 Ruijia's question(s) on integration

Roughly speaking Ruijia expressed general interest in integration to determine areas and volumes in or of Riemannian manifolds. The first comment is that in my introduction of the Riemannian metric tensor, and especially with respect to the example $\mathcal{B}$ the Riemannian metric was put forward as a construct that allowed the calculation of three things: lengths, angles, and areas. The calculation of lengths in particular required integration. Hopefully, the mechanics of this (first) integration for length are clear: If you want the length of a path Γ in a Riemannian manifold M, break the path up into disjoint pieces (or pieces with overlaps having zero length) so that each piece admits coordinates in a chart, that is, so that $\xi \circ \alpha$ parameterizes a path in a chart while α parameterizes the piece in Γ. Then integrate over the path(s) in the chart(s) using the appropriate integrand constructed using the metric coefficients. In this case

$$\langle (g_{ij})(\xi \circ \alpha)', (\xi \circ \alpha)' \rangle_{\mathbb{R}^n}^{1/2}.$$

Taking the length of a path Γ as an example of the "one-dimensional measure" of a subset of the manifold M, a similar procedure may be applied for higher dimensional measures of subsets of M. In each case, the metric tensor can be used to obtain the "correct" integrand(s) to use in the coordinate representation of small disjoint pieces in charts. You get the measures of the pieces and add them up to get the measure of the original subset of the manifold.

Ruijia asked a follow up question concerning whether or not the values calculated as described above agree with the Hausdorff measure(s). I'm assuming this means the manifold M is itself a submanifold of Euclidean space $\mathbb{R}^N$ so that Hausdorff measures

$$\mathcal{H}^k(S) = \alpha_k \sup_{\delta>0} \inf \left\{ \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam}(S_j)}{2} \right)^k : \operatorname{diam}(S_j) < \delta \right\} \tag{D.2}$$

of subsets S of M make sense. The inner infemum in (D.2) is taken over any countable covers $S_1, S_2, S_3, \ldots$ of S (by any sets), the constant $\alpha_k = \pi^{k/2}/\Gamma(k/2+1)$ is something like the area of the k sphere in $\mathbb{R}^{k+1}$, and the overall number $\mathcal{H}^k(S)$ is called the k-dimensional Hausdorff measure of S in $\mathbb{R}^N$. I think in this setting, the basic answer is that if S is a nice enough set to measure using the integration procedure mentioned above, then "yes" the answer you get from integrating is the same as the Hausdorff measure.

The explanation of why that is the answer is rather more complicated. Probably the best place to start thinking about that explanation is to consider some very simple cases. For example, what about $B_r(\mathbf{0}) \subset \mathbb{R}^n$? If you take the Hausdorff measure of this ball, do you get the usual formula for the measure of the ball obtained by integration? Next, one can ask about lower dimensional "balls" like

$$S = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) \in \mathbb{R}^n : \sum_{j=1}^{k} x_j^2 < r^2 \right\}.$$

Is $\mathcal{H}^k(S)$ what you would expect for the k-dimensional measure of a ball? Then one might think about spheres

$$S = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) \in \mathbb{R}^n : \sum_{j=1}^{k} x_j^2 = r^2 \right\}.$$

In this case, is it true that $\mathcal{H}^{k-1}(S) = \alpha_{k-1}$? The proofs of such results are found, generally speaking, in books which are known to be fairly difficult and some of which are notoriously difficult. The general subject in which these results are usually considered is geometric measure theory. The second section of Chapter 2 of [3] is devoted to showing $\mathcal{H}^n$ agrees with Lebesgue (outer) measure on $\mathbb{R}^n$. The proof of this assertion (predictably) usually relies somewhere on a relation between the Lebesgue measure of a set and the diameter of the set. Specifically, the fact that the Lebesgure measure of A is bounded above by

$$\alpha_n \left(\frac{\operatorname{diam}(A)}{2} \right)^n$$

is called the **isodiametric inequality**, and it is not so easy to prove. The other more complicated assertions of this sort are basically going to be implied by something called the **area formula**. Other references for the area formula with varying completeness and readability are [9], [7], and [4].

It seems to me that another direction to go with this question is to mention problems that centrally involve integration, namely the Willmore conjecture and Lawson's conjecture. It's nice that we now have enough Riemannian geometry to understand (at least roughly) the statements of these problems. Both involve a particular surface in the three-sphere

$$\mathbb{S}^3 = \{\mathbf{x} \in \mathbb{R}^4 : |\mathbf{x}| = 1\}.$$

This is a three-dimensional Riemannian manifold and contains a special submanifold (surface)

$$\mathcal{C} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2} \right\}$$

called the **Clifford torus**. In addition the entire class of conformal images of $\mathcal{C}$ is of interest for the Willmore conjecture.

Exercise D.1. Draw a picture of the stereographic projection of $\mathcal{C}$ in $\mathbb{R}^3$. Note that the north pole $\mathbf{e}_4 = (0, 0, 0, 1)$ is not in $\mathcal{C}$.

The Willmore conjecture is really about regular surfaces in $\mathbb{R}^3$. Let $\mathcal{S}$ be a compact regular surface in $\mathbb{R}^3$, then the **Willmore energy** is defined to be

$$W[\mathcal{S}] = \int_{\mathcal{S}} H^2$$

where H is the mean curvature of the surface $\mathcal{S}$. The important thing to know for the moment is that H is a real valued function on $\mathcal{S}$ and can be integrated.

Exercise D.2. Calculate the Willmore energy of a (round) sphere.

The Willmore conjecture (proved by Marques and Neves in 2014) states that if the surface $\mathcal{S}$ is a torus, then $W[\mathcal{S}] \geq 2\pi^2$ with the unique minimizers given by the conformal class of the (stereographic projection of the) Clifford torus.

Tom Willmore made this conjecture in 1965. Leon Simon proved the existence of a smooth minimizer in 1986.

The Clifford torus itself (considered as a submanifold of $\mathbb{S}^3 \subset \mathbb{R}^4$) is a **minimal surface**. This means that small enough pieces $\mathcal{S} \subset \mathcal{C}$ with smooth boundary Γ minimize the area functional

$$\text{area}[\mathcal{S}] = \int_{\mathcal{S}} 1 = \int_U \sqrt{g_{11}g_{22} - g_{12}^2} \tag{D.3}$$

over the admissible class of smooth surfaces $\mathcal{S}$ with $\partial\mathcal{S} = \Gamma$ fixed. In (D.3) $\mathbf{p} : U \to \mathcal{S}$ is a (global) chart function for the small piece $\mathcal{S}$ and the last equality can be taken as the definition.

Obviously the notion of minimality (or local area minimization) for surfaces, and the minimization of other integral functionals like the Willmore energy, is fundamentally about integration.

The Bryant duality theorem (1984) states roughly that there is a correspondence between critical surfaces for the Willmore functional and minimal surfaces in $\mathbb{S}^3$ via stereographic projection.

Lawson's conjecture (proved by Simon Brendle in 2012) is that the Clifford torus (and the conformal isometries or or rigid motions of it in $\mathbb{S}^3$) are the unique minimal surfaces in $\mathbb{S}^3$ which are topological tori.

Blaine Lawson made this conjecture in 1970.