Quiz 1: Riemannian surface

John McCuan

February 11, 2026

Let $\mathbb{R}^2$ denote two dimensional Euclidean space with the usual standard coordinates and structures. Let g_{11}, $g_{21} = g_{12}$, and g_{22} be three real numbers with the symmetric matrix (g_{ij}) positive definite.

Denote by $\widehat{\mathbb{R}^2} = (\mathbb{R}^2, (g_{ij}))$ the Euclidean plane with the inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = \sum_{i,j=1}^{2} g_{ij} a_i b_j$$

on each algebraic tangent space $S_{\mathbf{z}}\widehat{\mathbb{R}^2}$.

Preliminary problems

Problem 1 Define the length

$$\widehat{\text{length}}[\gamma]$$

of a curve $\gamma \in C^1([a,b] \to \widehat{\mathbb{R}^2})$. Here $C^1([a,b] \to \widehat{\mathbb{R}^2})$ denotes simply the trivial inclusion of $C^1([a,b] \to \mathbb{R}^2)$ into $C^0([a,b] \to \widehat{\mathbb{R}^2})$ with $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t)) \in S_{\gamma(t)}\widehat{\mathbb{R}^2}$.

Problem 2 Find the perimeter of the "unit" square

$$\{(u_1, u_2) \in \widehat{\mathbb{R}^2} : 0 < u_1, u_2 < 1\}.$$

Problem 3 Given $a, b > 0$, find the perimeter of the region

$$\left\{(u_1, u_2) \in \widehat{\mathbb{R}^2} : \frac{u_1^2}{a^2} + \frac{u_2^2}{b^2} < 1\right\}.$$

Problem 4 Define the area

$$\widehat{\text{area}}[\Omega]$$

of a region $\Omega \subset \widehat{\mathbb{R}^2}$

Problem 5 Find the area of the "unit" square from Problem 2, and find the area of the region from Problem 3.

Main problem

Main Problem 1 Find a surface $\mathcal{S} = X(\mathbb{R}^2)$ given by a single parameterization

$$X : \mathbb{R}^2 \to \mathbb{R}^3$$

for which all the lengths and areas are actually (as computed using the standard Euclidean structures in $\mathbb{R}^3$) those of the corresponding regions in $\widehat{\mathbb{R}^2}$.